# 6.2 Concepts Worksheet Continued NAME KEY!

**Example B:**

Briefly outlined:

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{k^3}{n^4} = \lim_{n\to\infty}\sum_{k=1}^{2n}\left[\frac{k^3}{n^3}\cdot\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\sum_{k=1}^{2n}\left[\left(\frac{k}{n}\right)^3\cdot\frac{1}{n}\right]$$

Step 1: $dx = \frac{1}{n}$

Step 2: $x$-values: $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{2n}{n}$ $\left(x = \frac{k}{n}\right)$

Step 3: Limit of $x$-values: $\lim_{n\to\infty}\frac{1}{n} = 0$; $\lim_{n\to\infty}\frac{2n}{n} = 2$

Step 4: $dx = \frac{2-0}{2n} = \frac{1}{n}$, which agrees with $dx$ chosen in Step 1.

Step 5: $f(x) = x^3$

Therefore $\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{k^3}{n^4} = \int_0^2 x^3\,dx$

Express the following Riemann Sums as definite integrals:

1. $\lim_{n\to\infty}\left[\frac{1}{n^3}+\frac{4}{n^3}+\frac{9}{n^3}+\cdots+\frac{n^2}{n^3}\right] = \int_0^1 x^2\,dx$

2. $\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{k}{n}+\left(\frac{k}{n}\right)^2\right]\frac{1}{n} = \int_0^1 (x+x^2)\,dx$

3. $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}\left(1+\frac{2k}{n}\right)} = \int_0^1 \sqrt{1+2x}\,dx$ $\boxed{\text{if } x = k/n}$

OR $\frac{1}{2}\int_0^2 \sqrt{1+x}\,dx$ $\boxed{\text{if } x = \frac{2k}{n}}$

OR $\frac{1}{2}\int_1^3 \sqrt{x}\,dx$, $\boxed{\text{if } x = 1+\frac{2k}{n}}$

4. $\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{2k} = \int_1^2 \frac{1}{2}x^{-1}\,dx$ ; if $x = k/n$

OR $\frac{1}{2}\int_0^1 (1+x)^{-1}\,dx$; if $x = \frac{2k}{n}$

OR $\frac{1}{2}\int_2^4 \frac{1}{x}\,dx$ ; if $x = \frac{2k}{n}$

5. $\lim_{n\to\infty}\sum_{k=1}^{2n}\left[\frac{1}{1+\frac{2k}{n}}\cdot\frac{1}{n}\right]$

$= \int_0^2 \frac{1}{1+2x}\,dx$

## 6.2 Concepts Worksheet Continued NAME______________________

### Concept Connectors

Based on problem 5 above, use the following choices for $x$-values and define an appropriate integral.

6. $x = \frac{k}{n}$ $\int_0^2 \left(\frac{1}{1+2x}\right) dx$

7. $x = \frac{2k}{n}$ $\frac{1}{2}\int_0^4 \left(\frac{1}{1+x}\right) dx$

8. $x = 1 + \frac{2k}{n}$ $\frac{1}{2}\int_1^5 \frac{1}{x}\, dx$

9. Do the integrals you wrote in problems 6–8 all have the same value? Explain.

Yes! All are equal because it is the same area under the curve.

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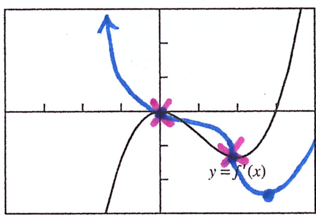
## 6.3–6.4 Concepts Worksheet

NAME________________________________

### Graphical Antidifferentiation

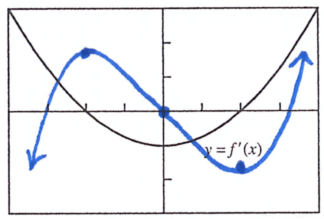
Each of the following graphs represents the derivative of a continuous function $f$. Sketch a possible graph of $y = f(x)$ on the same set of axes as the derivative, assuming $f(0) = 0$.

**1.**



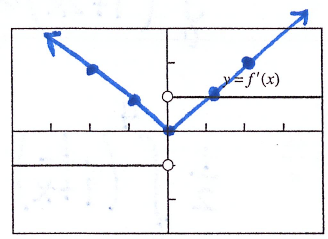

[–4, 4] by [–3, 3]

**2.**



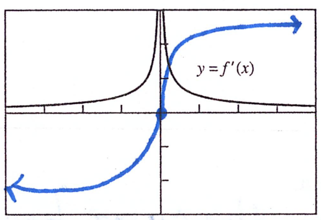

[–4, 4] by [–3, 3]

**3.**



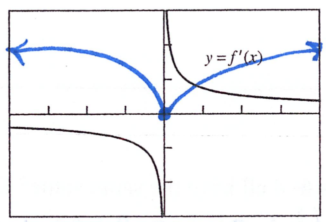

[–4, 4] by [–3, 3]

**4.**




[–4, 4] by [–3, 3]

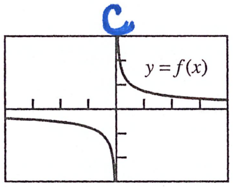
**5.**




[–4, 4] by [–3, 3]

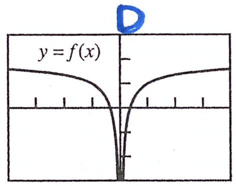
The following graphs of $f'(x)$ involve nonexistent derivatives at $x = 0$, because $f(x)$ is discontinuous at $x = 0$. Match each graph of $f(x)$ to the corresponding $f'(x)$ graph.

**6.** C



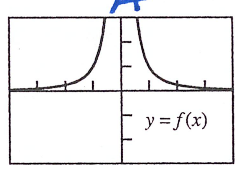

[–4, 4] by [–3, 3]

**7.** D



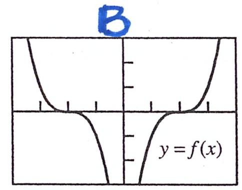

[–4, 4] by [–3, 3]

**8.** A



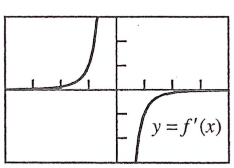

[–4, 4] by [–3, 3]

**9.** B



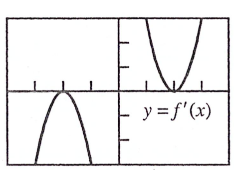

[–4, 4] by [–3, 3]

**(a)**



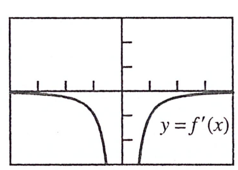

[–4, 4] by [–3, 3]

**(b)**



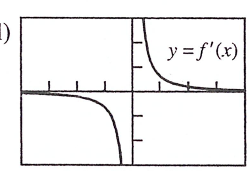

[–4, 4] by [–3, 3]

**(c)**




[–4, 4] by [–3, 3]

**(d)**



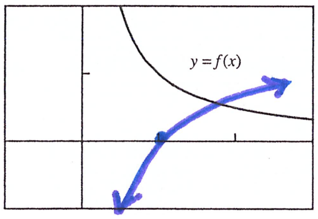

[–4, 4] by [–3, 3]

# 6.3–6.4 Concepts Worksheet Continued    NAME____________________

Given a graph of a function $f$, what would the graph of the function $F(x) = \int_a^x f(t)\,dt$ look like? The following questions should lead you to a rough shape of the graph of $y = F(x)$.

**10.** Let $f$ be the function whose graph is shown below, and let $F(x) = \int_1^x f(t)\,dt$, for $x > 0$:

Area under the curve.




[−1, 3] by [−1, 2]

$F(0) = \int_1^0 f(t)dt = -\int_0^1 f(t)dt = (-)$

$F(1) = \int_1^1 f(t)dt = 0$

$F'(x) = \frac{d}{dx}\int_1^x f(t)dt = f(x)$ → always (+) ∴ F(x) is always incr.

(a) Evaluate $F(1)$. $= 0$

(b) $F(x)$ is always increasing
increasing/decreasing

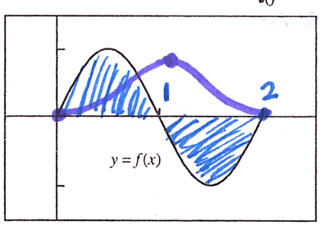
(c) $F(x)$ is negative for what $x$ values? $0 < x < 1$

(d) State any maximum or minimum points of $F(x)$. None (always incr.)

$F''(x) = f'(x)$ → cc down, (−)

(e) Draw a rough sketch of the integral function, $F(x)$, on the graph above.

**11.** Let $f$ be the function whose graph is shown below, where $f$ is defined for $0 \le x \le 2$ and has point symmetry about (1, 0). Let $F(x) = \int_0^x f(t)\,dt$




[−0.5, 2.5] by [−1.5, 1.5]

$F(0) = \int_0^0 f(t)dt = 0$  $F(2) = \int_0^2 f(t)dt = 0$ ((+) & (−) cancels out)

$F'(x) = \frac{d}{dx}\int_0^x f(t)dt = f(x)$

$F''(x) = f'(x)$

$F'(x)$: 0 | + | 1 | − | 2

$F''(x)$: 0 | + | $\frac{1}{2}$ | − | $\frac{3}{2}$ | + | 2

(a) Evaluate $F(0)$. $= 0$

(b) Evaluate $F(2)$. $= 0$

(c) $F(x)$ has a maximum value at $x =$ 1

(d) $F'(x)$ has a maximum value at $x =$ $\frac{1}{2}$

(e) $F'(x)$ has a minimum value at $x =$ $\frac{3}{2}$

(f) $F''(x) > 0$ for what values of $x$? $0 < x < \frac{1}{2}$ & $\frac{3}{2} < x < 2$

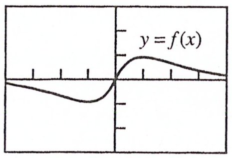
(g) $F''(x) < 0$ for what values of $x$? $\frac{1}{2} < x < \frac{3}{2}$

(h) Draw a rough sketch of the integral function, $F(x)$, on the graph above.

## 6.3–6.4 Concepts Worksheet Continued NAME____________

### Concept Connectors

The graph of an odd function $f$ is shown. Let $F(x) = \int_0^x f(t)\,dt$ and assume that $F(a) = b$.




[–4, 4] by [–3, 3]

**12.** Evaluate each definite integral.

(a) $\int_{-a}^{a} f(t)\,dt$ $= 0$ (it cancels out)

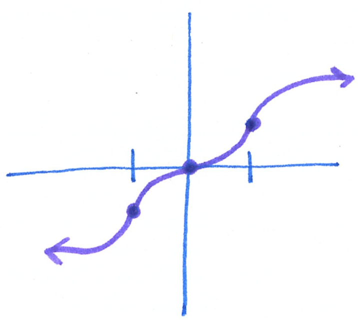
(b) $\int_{-a}^{a} |f(t)|\,dt$ $= 2b$ (twice the area)

(c) $\int_{-a}^{a} f(|t|)\,dt$ $= 2b$ (twice the area)

$F'(x) = f(x)$
$F''(x) = f'(x)$

**13.** Draw a rough sketch of the integral function $\underbrace{F(x) = \int_0^x f(t)\,dt}$ on the same set of axes of the graph of $y = f(x)$.

area under the curve from 0 to x.



$\int_0^x f(t)\,dt = -\int_x^0 f(t)\,dt$

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## 6.4 Concepts Worksheet

NAME________________

### The Fundamental Theorem of Calculus

Test your understanding of parts 1 and 2 of the Fundamental Theorem by simplifying the following:

1. If $F(x)=\int_1^x \frac{dt}{1+t^2}\,dt$, then $F'(x)=$ $\boxed{\frac{1}{1+x^2}}$

2. If $F(x)=\int_3^{2x}\sqrt{t^2+1}\,dt$, then $F'(x)=$ $\left(\sqrt{(2x)^2+1}\right)\cdot 2=\boxed{2\sqrt{4x^2+1}}$

3. If $F(x)=\int_x^{2x}\frac{1}{t}\,dt$, $x>0$, then $F'(x)=$ $\int_0^{2x}-\int_0^{x}=\frac{1}{2x}\cdot 2-\frac{1}{x}=\frac{1}{x}-\frac{1}{x}=\boxed{0}$

4. If $F(x)=\int_0^{\sqrt{x}}\sin(t^2)\,dt$, then $F'(x)=$ $\sin(\sqrt{x})^2\left[\frac{1}{2}x^{-1/2}\right]=\boxed{\frac{\sin x}{2\sqrt{x}}}$

5. If $x>0$, then $\frac{d}{dx}\int_1^{1/x}\frac{2}{t}\,dt=$ $\frac{2}{1/x}\left(-x^{-2}\right)=\frac{-2x}{x^2}=\boxed{\frac{-2}{x}}$

6. $\int_1^4\left(\frac{d}{dx}\sqrt{x^2-1}\right)dx=$ $\sqrt{(4)^2-1}-\sqrt{(1)^2-1}=\sqrt{15}-0=\boxed{\sqrt{15}}$

derivative of the integral @ 4 and 1.

### Concept Connectors

Using the definition of a derivative and the Fundamental Theorem of Calculus, evaluate the following:

7. $\lim_{h\to 0}\frac{\int_2^{2+h}\sqrt{x^2+x}\,dx}{h}=$ $f'(2)=\sqrt{(2)^2+(2)}=\boxed{\sqrt{6}}$

Evaluate @ x=2

8. $\lim_{x\to 1}\frac{\int_1^x \frac{1}{t^4+1}\,dt}{x-1}=$ $f'(1)=\frac{\frac{d}{dx}\int^x \frac{1}{t^4+1}\,dt}{\frac{d}{dx}(x-1)}=\frac{\frac{1}{x^4+1}}{1}=\frac{1}{x^4+1}\rightarrow\frac{1}{(1)^4+1}=\boxed{\frac{1}{2}}$

Evaluate at x=1

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## 6.5 Concepts Worksheet

NAME________________________

**Definite Integral Approximations**

1. Determine and evaluate a definite integral for which $\frac{1}{40}[(0)^3 + 2\cdot(0.05)^3 + 2(0.1)^3 + \cdots + 2(1.95)^3 + (2)^3]$ is a trapezoidal approximation. Which is greater, the integral or trapezoidal approximation? Why?

$\int_0^2 x^3 dx = 4$ since $x^3$ is cc up from 0 to 2, the approximation will be greater

$\frac{1}{2}\left[\frac{2}{4} + 2\left(\frac{3}{4}\right) + 2\left(\frac{4}{4}\right) + \ldots + 2\left(\frac{19}{4}\right) + \frac{20}{4}\right]$

2. Determine and evaluate a definite integral for which $\frac{1}{2}\left[\frac{1}{2} + 1.5 + 2 + 2.5 + 3 + \cdots + 9.5 + 5\right]$ is a trapezoidal approximation. Which is greater, the integral or the trapezoidal approximation? Why?

$\int_2^{20} \frac{x}{4} dx = \frac{99}{2} = 49.5$ since $\frac{x}{4}$ is a straight line, the approx. is exactly correct.

or $\int_{1/2}^{5} 4x\, dx$

3. (a) Determine and evaluate a definite integral for which $\frac{1}{6}\left[0 + 4\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 4\left(\frac{\sqrt{6}}{2}\right) + 2\sqrt{2} + 4\left(\frac{\sqrt{10}}{2}\right) + 2\sqrt{3} + 4\left(\frac{\sqrt{14}}{2}\right) + 2\right]$ is a Simpson's rule approximation.

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(b) Set up the trapezoidal approximation for the integral answer from (a) using the same number of partitions.

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(c) Compare the Simpson's Rule and trapezoidal approximations geometrically. Which do you think should be larger? Why?

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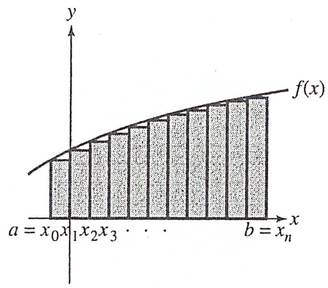
### Concept Connectors

4. For a strictly increasing positive-valued function over an interval $[a, b]$ explain why the average of the sums of the areas of $n$ inscribed and circumscribed rectangles, all of constant width, is the trapezoidal approximation. (Note: For a strictly decreasing function over an interval, a similar conclusion can be obtained.)

$$\Delta x = \frac{b-a}{n}$$



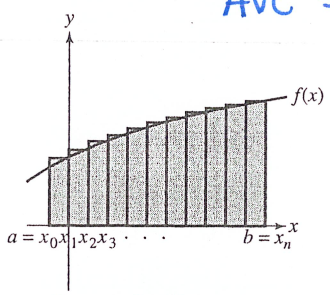
Inscribed Rectangles

$$I = [f(x_0)+f(x_1)+f(x_2)+\ldots+f(x_{n-1})]\cdot\Delta x$$

$$C = [f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_n)]\cdot\Delta x$$

$$Ave = \frac{I+C}{2}$$

$$Ave = \frac{[f(x_0)+f(x_1)+\ldots+f(x_{n-1})]\cdot\Delta x+[f(x_1)+f(x_2)+\ldots+f(x_{n-1})+f(x_n)]\cdot\Delta x}{2}$$



Circumscribed Rectangles

$$= \frac{\Delta x\,[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)]}{2}$$

$$= \frac{1}{2}\left(\frac{b-a}{n}\right)[f(x_0)+2f(x_1)+\ldots+2f(x_{n-1})+f(x_n)]$$

$$\boxed{= \frac{b-a}{2n}[f(x_0)+2f(x_1)+\ldots+2f(x_{n-1})+f(x_n)]}$$