Rates of Change & Limits (Section 2.1)

* Average Speed: $\dfrac{\text{change in distance } (y)}{\text{change in time } (t)} = \dfrac{\Delta y}{\Delta t}$

little bit

* Instantaneous Speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{f(t+h) - f(t)}{h} \rightarrow \lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$

ex: $y = 16t^2$ Find the ave. speed from $t=0$ to $t=3$ sec.

ave speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{f(3) - f(0)}{3-0} = \dfrac{16(3)^2 - \cancel{16(0)^2}}{3} = \dfrac{144}{3} = \boxed{48 \text{ units/sec}}$

0

Find the inst. speed @ $t=2$ sec.

inst. speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{f(2+h) - f(2)}{h} = \dfrac{16(2+h)^2 - 16(2)^2}{h}$

$= \dfrac{16(4+4h+h^2) - 64}{h} = \dfrac{\cancel{64} + 64h + 16h^2 - \cancel{64}}{h}$

$= \dfrac{\cancel{h}(64+16h)}{\cancel{h}} = 64+16h \rightarrow \lim_{h \to 0}(64+16h)$

$\boxed{= 64 \text{ units/sec}}$

* Definition of a Limit:

The function $f(x)$ has a limit (L) as x approaches "c", if given any positive # (ε), there is a positive # (δ) such that: $0 < |x-c| < \delta \rightarrow |f(x) - L| < \varepsilon$

** We write: $\lim_{x \to c} f(x) = L$ ← the limit is a y-value!

* Properties of Limits:

- constant value: $\lim_{x\to c} k = k$
- identity: $\lim_{x\to c} x = c$

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then...

- sum: $\lim_{x\to c} [f(x)+g(x)] = L+M$
- difference: $\lim_{x\to c} [f(x)-g(x)] = L-M$
- product: $\lim_{x\to c} [f(x)\cdot g(x)] = L\cdot M$
- quotient: $\lim_{x\to c} [f(x)/g(x)] = L/M$; $M\neq 0$
- constant multiple: $\lim_{x\to c} [k\cdot f(x)] = k\cdot L$

** $\lim_{x\to 0} \frac{\sin x}{x} = 1$ ← Memorize This!

ex: $\lim_{x\to 2} (x^2+x-4) = (2)^2+2-4$ $\boxed{=2}$

ex: $\lim_{x\to c} \left(\frac{4x^3-1}{x+1}\right) = \boxed{\frac{4c^3-1}{c+1} ;\ c\neq -1}$

ex: $\lim_{x\to 0} \left(\frac{\tan x}{x}\right) = \lim_{x\to 0} \left(\frac{\frac{\sin x}{\cos x}}{\frac{x}{1}}\right) = \lim_{x\to 0} \left(\frac{\sin x}{\cos x}\cdot\frac{1}{x}\right) = \lim_{x\to 0} \left(\cancel{\frac{\sin x}{x}}^{\,1}\cdot\frac{1}{\cos x}\right)$

$= \lim_{x\to 0} \left(1\cdot\frac{1}{\cos x}\right) = \frac{1}{\cos 0} = \frac{1}{1} \boxed{=1}$

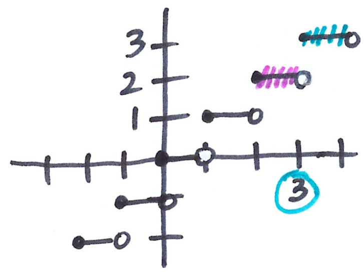
ex: $\lim_{x\to 7} \left(\frac{x^2-49}{x-7}\right) = \lim_{x\to 7} \frac{(x+7)\cancel{(x-7)}}{\cancel{(x-7)}} = \lim_{x\to 7} (x+7) = 7+7 \boxed{=14}$

* **Two-Sided Limits:** Right-hand limit: $\lim_{x \to c^+} f(x)$

Left-hand limit: $\lim_{x \to c^-} f(x)$

These two limits **MUST** be equal for the $\lim_{x \to c} f(x) = L$

ex: $\lim_{x \to 3} (\text{int } x)$

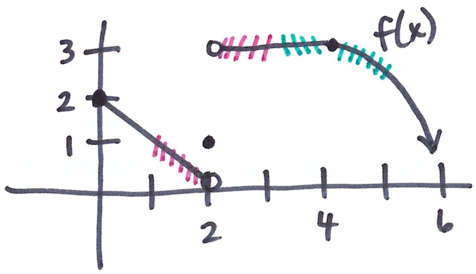


$\lim_{x \to 3^+} (\text{int } x) = 3$

$\lim_{x \to 3^-} (\text{int } x) = 2$

$\therefore \lim_{x \to 3} (\text{int } x) = \underline{\underline{\text{DNE}}}$

ex:



$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2^-} f(x) = 0$

$\lim_{x \to 2} f(x) = \text{DNE}$

eventhough $f(2) = 1$

$\lim_{x \to 4^+} f(x) = 3$

$\lim_{x \to 4^-} f(x) = 3$

$\lim_{x \to 4} f(x) = 3$