Chain Rule (Section 4.1)

* If $f(x)$ & $g(x)$ are differentiable @ x, then the derivative of

$$f\circ g \rightarrow \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\cdot g'(x)$$

OR If $y = f(u)$ & $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$$\downarrow$$

$$y' = f'(u)\cdot g'(x)$$

ex: $y = \sin(3x+1)$ $\quad u = 3x+1$, $du = 3$

$y = \sin u$

$y' = \cos u \cdot du$

$y' = \cos(3x+1)\cdot 3 \boxed{= 3\cos(3x+1)}$

ex: $y = \sin^2 x = (\sin x)^2$

$u = \sin x$ $\quad y = u^2$

$du = \cos x$ $\quad y' = 2u\cdot du$

$\boxed{y' = 2(\sin x)(\cos x)}$

OR $y = (\sin x)^2$

$y' = 2(\sin x)^1(\cos x)$

ex: $y = \sin(x^2)$ $\quad u = x^2$, $du = 2x$

$y = \sin u$

$y' = \cos u \cdot du$

$y' = \cos(x^2)\cdot 2x$

$\boxed{y' = 2x\cos(x^2)}$

OR $y = \cos(x^2)\cdot 2x$

$y = 2x\cos(x^2)$

ex: $y = \frac{1}{\sqrt{\sin x}} = (\sin x)^{-1/2}$

$u = \sin x$ $\quad y = u^{-1/2}$

$du = \cos x$ $\quad y' = -\frac{1}{2}u^{-3/2}\cdot du$

$y' = -\frac{1}{2}(\sin x)^{-3/2}(\cos x)$

$\boxed{y' = \frac{-\cos x}{2(\sin x)^{3/2}}}$

OR $y = (\sin x)^{-1/2}$

$y' = -\frac{1}{2}(\sin x)^{-3/2}\cdot\cos x$

$y' = \frac{-\cos x}{2(\sin x)^{3/2}}$

ex: $y = (x^3-1)^4$

$u = x^3-1$ $\quad y = u^4$

$du = 3x^2$ $\quad y' = 4u^3\cdot du$

$y' = 4(x^3-1)^3\cdot 3x^2$

$\boxed{y' = 12x^2(x^3-1)^3}$

OR $y' = 4(x^3-1)^3(3x^2)$

$= 12x^2(x^3-1)^3$

ex: $y = \sec(4x)$

$u = 4x$ $\quad y = \sec u$

$du = 4$

$y' = \sec u\cdot\tan u\cdot du$

$y' = \sec(4x)\cdot\tan(4x)\cdot 4$

$\boxed{y' = 4\sec(4x)\tan(4x)}$

OR $y' = \sec(4x)\tan(4x)\cdot 4$

$y' = 4\sec(4x)\tan(4x)$

ex: $y = \dfrac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$

$y' = x\left[-\frac{1}{2}(x^2+1)^{-3/2}\cdot(2x)\right] + (x^2+1)^{-1/2}\cdot 1$

in common!

$y' = (x^2+1)^{-3/2}\left[-\cancel{x^2} + (\cancel{x^2}+1)\right]$

$$\boxed{y' = \frac{1}{(x^2+1)^{3/2}}}$$

Quotient Rule $y = \dfrac{x}{(x^2+1)^{1/2}}$

$$y' = \frac{(x^2+1)^{1/2}(1) - x\left[\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x\right]}{\left[(x^2+1)^{1/2}\right]^2}$$

$$y' = \frac{(x^2+1)^{-1/2}\left[(\cancel{x^2}+1)^1 - \cancel{x^2}\right]}{(x^2+1)^1} = \frac{1}{(x^2+1)^{3/2}}$$

ex: $y = \sqrt{\dfrac{x}{x^2+1}}$ $\quad u = \dfrac{x}{x^2+1}$ $\quad du = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{(x^2+1)^2}$

$y = \sqrt{u} = u^{1/2}$ $\quad du = \dfrac{-x^2+1}{(x^2+1)^2}$

$y' = \frac{1}{2}u^{-1/2}\cdot du$

$y' = \frac{1}{2}\left(\frac{x}{x^2+1}\right)^{-1/2}\cdot\frac{-x^2+1}{(x^2+1)^2} = \frac{1}{2}\cdot\frac{\cancel{(x^2+1)^{1/2}}}{x^{1/2}}\cdot\frac{(-x^2+1)}{(x^2+1)^{\cancel{2}\ 3/2}}$

$$\boxed{y' = \frac{-x^2+1}{2x^{1/2}(x^2+1)^{3/2}}}$$

#33 $f(u) = u^5+1$ $\quad u = \sqrt{x} = x^{1/2}$

$f'(u) = 5u^4\cdot du$ $\quad du = \frac{1}{2}x^{-1/2}$

$= 5(x^{1/2})^4\left[\frac{1}{2}x^{-1/2}\right]$

$= \dfrac{5x^2}{2x^{1/2}} = \boxed{\dfrac{5x^{3/2}}{2}}$

ex: $y = x^2\tan^2(4x) = x^2(\tan 4x)^2$

$u = \tan 4x$ $\quad y = x^2\cdot u^2$

$du = 4\sec^2 4x$ $\quad y' = x^2(2u\,du) + u^2(2x)$

$y' = x^2\left[2(4\sec^2 4x)(\tan 4x)\right] + \tan^2 4x(2x)$

$y' = 8x^2\sec^2 4x\tan 4x + 2x\tan^2 4x$

$$\boxed{y' = 2x\tan 4x\,(4x\sec^2 4x + \tan 4x)}$$

** Parametric: x(t) & y(t) equations

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

#41 $x = 2\cos t$ $y = 2\sin t$ Find the tangent line @ $t = \pi/4$.

Point $\left[\begin{array}{l} x(\pi/4) = 2\cos(\pi/4) = \cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right) = \sqrt{2} \\ y(\pi/4) = 2\sin(\pi/4) = \cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right) = \sqrt{2} \end{array}\right.$

Point: $(\sqrt{2}, \sqrt{2})$

Slope: $m = -1$

$$\boxed{\therefore\ y - \sqrt{2} = -1(x - \sqrt{2})}$$

slope: $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{2}\cos t}{-\cancel{2}\sin t} = -\cot(t)$

@ $t = \pi/4$ $m = -\cot(\pi/4)$

$m = -1$

#56

x	f(x)	g(x)	f'(x)	g'(x)
2	8	2	1/3	-3
3	3	-4	2π	5

a) $\frac{d}{dx}[2 \cdot f(x)] = 2f'(x)$

@ $x = 2$ $= 2f'(2) = 2(1/3) = \boxed{2/3}$

c) $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x)\, f'(x)$

@ $x = 3$ $= f(3) \cdot g'(3) + g(3) \cdot f'(3)$

$3 \cdot 5 + -4 \cdot 2\pi$

$\boxed{= 15 - 8\pi}$

e) $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

@ $x = 2$ $= f'(g(2)) \cdot g'(2)$

$= f'(2) \cdot -3$

$= (1/3)(-3) \boxed{= -1}$

g) $\frac{d}{dx}\left(\frac{1}{g^2}\right) = \frac{d}{dx}(g^{-2}) = -2g^{-3} \cdot g'$

@ $x = 3$ $= -2[g(3)]^{-3} \cdot g'(3)$

$= -2(-4)^{-3}(5)$

$= \frac{-10}{(-4)^3} = \frac{10}{64} = \boxed{\frac{5}{32}}$

*More examples of Chain Rule:

ex: $y = \sin(\sqrt{x})$

$u = \sqrt{x} = x^{1/2}$ $\quad y = \sin u$

$du = \frac{1}{2}x^{-1/2}$ $\quad y' = \cos u \cdot du$

$y' = \cos(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$

$$\boxed{y' = \frac{\cos(\sqrt{x})}{2\sqrt{x}}}$$

ex: $y = \frac{6}{\sqrt{x^2+4}} = 6(x^2+4)^{-1/2}$ NOT QUOTIENT RULE!

$y' = 6\left[-\frac{1}{2}(x^2+4)^{-3/2}(2x)\right]$

$$\boxed{y' = \frac{6x}{(x^2+4)^{3/2}}}$$

ex: $y = \tan(3x) - \csc(4x)$

$y' = \sec^2(3x) \cdot 3 + [+\csc(4x)\cot(4x) \cdot 4]$

$$\boxed{y' = 3\sec^2(3x) + 4\csc(4x)\cot(4x)}$$

ex: