Definite Integrals & Antiderivatives (Section 6.3)

* Rules:

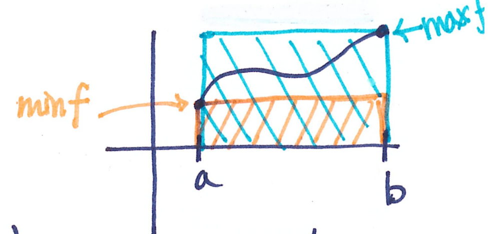
① $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

② $\int_a^a f(x)\,dx = 0$

③ $\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$

④ $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

⑤ $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



⑥ $\underbrace{\min f\cdot(b-a)}_{\text{this area is less}} \le \underbrace{\int_a^b f(x)\,dx}_{\text{actual area}} \le \underbrace{\max f\cdot(b-a)}_{\text{this area is more}}$ → example

⑦ $f(x) \ge g(x)$ on $[a,b] \rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a,b] \rightarrow \int_a^b f(x)\,dx \ge 0$

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ex: If $\int_{-2}^{3} f(x)\,dx = 4$; $\int_3^5 f(x)\,dx = -2$; $\int_{-2}^{3} h(x)\,dx = -1$, then find:

① $\int_5^3 f(x)\,dx = -\int_3^5 f(x)\,dx$

$= -(-2) \boxed{= 2}$

② $\int_{-2}^{5} f(x)\,dx = \int_{-2}^{3} f(x)\,dx + \int_3^5 f(x)\,dx$

$= 4 + (-2) \boxed{= 2}$

③ $\int_3^5 3f(x)\,dx = 3\int_3^5 f(x)\,dx$

$= 3(-2) \boxed{= -6}$

④ $\int_{-2}^{3} [2f(x) + 3h(x)]\,dx$

$= 2\int_{-2}^{3} f(x)\,dx + 3\int_{-2}^{3} h(x)\,dx$

$= 2(4) + 3(-1) = 8 - 3 \boxed{= 5}$

* **Average Value:** If f is integrable on $[a,b]$, it's average value or (mean value) on $[a,b]$ is:

$$\boxed{\text{ave } f = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

ex: Find the average value of $f(x) = 9 - x^2$ on $[0,4]$.

$$\text{Ave Value} = \frac{1}{4-0}\int_0^4 (9-x^2)\,dx \longrightarrow \text{Can use } \underbrace{\text{Math \#9}}_{\text{fnInt}:}\ \int_{\boxed{0}}^{\boxed{4}} (\boxed{y_1})\,d\boxed{x} \quad y_1 = 9-x^2$$

$$= 14.\overline{6}$$

$$\tfrac{1}{4}(14.\overline{6}) = 3.\overline{6}$$

Antiderivative of $9-x^2 = 9x - \frac{x^3}{3}$

$$= \frac{1}{4}\left(9x - \frac{x^3}{3}\right)\Big|_0^4$$

*Now plug in the upper-lower bound.

$$= \frac{1}{4}\left[9(4) - \frac{(4)^3}{3} - \cancel{\left(9\cdot 0 - \frac{0^3}{3}\right)}\right]$$

$$= \frac{1}{4}\left(36 - \frac{64}{3}\right)$$

$$= \frac{1}{4}\left(\frac{44}{3}\right) = \frac{11}{3} = \boxed{3.\overline{6}}$$

Same!

** **Mean Value Theorem for Definite Integrals:**

If f is continuous on $[a,b]$, then at some point "c" in $[a,b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

ex: Find the average value of $y = -\frac{x^2}{2}$ on $[0,3]$. At what point does that occur?

$$\text{Ave Value} = \frac{1}{3-0}\int_0^3 -\frac{x^2}{2}\,dx = \frac{1}{3}\left(-\frac{x^3}{6}\right)\Big|_0^3$$

$$= -\frac{1}{18}\left(3^3 - 0^3\right)$$

$$= -\frac{27}{18} = \boxed{-3/2}$$

$$f(c) = -\frac{3}{2}$$

$$\downarrow$$

$$-\frac{c^2}{2} = -\frac{3}{2}$$

$$c^2 = 3$$

$c = \pm\sqrt{3} \longrightarrow$ must be on $[0,3]$ $\therefore$ "c" must equal $\sqrt{3} \rightarrow \boxed{c = \sqrt{3}}$

ex: $\int_0^{\pi} \underbrace{\sin x}\, dx = -\cos x \Big|_0^{\pi}$ (plug in upper−lower)

antiderivative of $\sin x = -\cos x$

$= -\cos\pi + (+\cos 0)$

$= -(-1) + 1$

$\boxed{= 2}$

ex: $\int_0^2 4x\, dx = \frac{4x^2}{2} \Big|_0^2$

antideriv. of $4x = \frac{4x^2}{2}$

$= 2x^2 \Big|_0^2$

$= 2(2^2 - 0^2)$

$\boxed{= 8}$

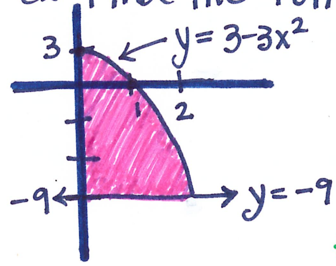
ex: $\int_{-1}^{3} (2t-3)\, dt = \frac{2t^2}{2} - 3t \Big|_{-1}^{3}$

$= t^2 - 3t \Big|_{-1}^{3}$

$= [3^2 - 3(3)] - [(-1)^2 - 3(-1)]$

$= \cancel{(9-9)}^{\,0} - (1+3) \boxed{= -4}$

ex: Find the TOTAL shaded area of:



* because this is above & below the axis, you must do this differently!

* Best way: move the graph up 9 units so it is above the x-axis. → $y = (3-3x^2)+9$

$y = 12 - 3x^2$

$\int_0^2 (12-3x^2)dx = 12x - \frac{3x^3}{3} \Big|_0^2$

$= 12x - x^3 \Big|_0^2$

$= [12(2) - (2)^3] - [12(0) - (0)^3]$

$= 24 - 8 \boxed{= 16}$

ex: If $y = -x^2 + 5x - 4$, integrate the function over $[0,2]$.

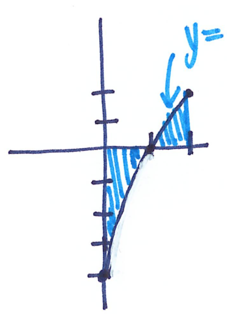
a) $\int_0^2 (-x^2 + 5x - 4)\, dx$

$= \frac{-x^3}{3} + \frac{5x^2}{2} - 4x \Big|_0^2$

$= \left[\frac{-2^3}{3} + \frac{5(2)^2}{2} - 4(2)\right] - [0]$

$= -\frac{8}{3} + 10 - 8 = \boxed{-2/3}$

b) Find the TOTAL area of the region between the graph & x-axis.



seperate this for TOTAL area!

$-\int_0^1 (-x^2 + 5x - 4)dx + \int_1^2 (-x^2 + 5x - 4)dx$

$= \frac{x^3}{3} - \frac{5x^2}{2} + 4x \Big|_0^1 + \frac{-x^3}{3} + \frac{5x^2}{2} - 4x \Big|_1^2$

$= \left(\frac{1}{3} - \frac{5}{2} + 4\right) - (0) + \left(\frac{-2^3}{3} + \frac{5(2)^2}{2} - 4(2)\right) + \left(\frac{+1}{3} - \frac{5}{2} + 4\right)$

$= \frac{2}{3} - \frac{10}{2} + \cancel{8} - \frac{8}{3} + \frac{20}{2} - \cancel{8}$

$\boxed{= 3}$

More definite integrals:

ex: $\int_0^{\pi/4} \sec^2 x\,dx = \tan x \Big|_0^{\pi/4}$

$= \tan \pi/4 - \tan 0$

$= 1-0$

$\boxed{=1}$

ex: $\int_{-2}^{2} e^x\,dx = e^x \Big|_{-2}^{2}$

$= e^2 - e^{-2}$

$\boxed{= e^2 - \frac{1}{e^2}}$

ex: $\int_1^2 \frac{5}{x^2}\,dx = \int_1^2 5x^{-2}dx$

$= 5(-x^{-1}) \Big|_1^2$

$= -5\left(\frac{1}{x}\right) \Big|_1^2$

$= -5\left(\frac{1}{2} - \frac{1}{1}\right)$

$= -5\left(\frac{-1}{2}\right)$ $\boxed{=\frac{5}{2}}$

ex: $\int_1^e \frac{3}{x}\,dx = 3\int_1^e \frac{1}{x}\,dx$

$= 3 \ln x \Big|_1^e$

$= 3(\underbrace{\ln e}_{1} - \underbrace{\ln 1}_{0})$

$= 3(1-0)$

$\boxed{=3}$

ex: $\int_0^1 \frac{4}{1+x^2}\,dx = 4\int_0^1 \frac{1}{1+x^2}dx$

$= 4(\tan^{-1}x) \Big|_0^1$

$= 4(\tan^{-1} 1 - \tan^{-1} 0)$

$= 4(\pi/4 - 0)$

$\boxed{= \pi}$

ex: $\int_0^{\pi/3} \sec x \tan x\,dx$

$= \sec x \Big|_0^{\pi/3}$

$= \sec \pi/3 - \sec 0$

$= 2-1$

$\boxed{=1}$