Fundamental Theorem of Calculus (Section 6.4)

* **FTC, Part 1**: If f is continuous on $[a,b]$, then the function $F(x)=\int_a^x f(t)\,dt$ has a derivative at every point x in $[a,b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)\cdot\frac{dx}{dx}$$

← This is the derivative of the function/variable you plugged in!

The constant must be the lower bound!

$$\boxed{** \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)\cdot x'}$$

ex: $\frac{d}{dx}\int_\pi^{2x}(\cos t)\,dt = \cos(2x)\cdot 2$

$\boxed{= 2\cos(2x)}$

ex: $\frac{d}{dx}\int_{x^2}^{5}(3t\cdot \sin t)\,dt$

$= \frac{d}{dx} = -\int_5^{x^2}(3t\cdot \sin t)\,dt$

$= -\left[3x^2\cdot \sin(x^2)\cdot 2x\right]$

$\boxed{= -6x^3\sin(x^2)}$

* **FTC, Part 2**: If f is continuous at every point of $[a,b]$ and if $F(x)$ is any antiderivative of f on $[a,b]$ then:

$$\int_a^b f(x)\,dx = \underbrace{F(b)-F(a)}_{\text{upper bound}\,-\,\text{lower bound}}$$

← This finds the "signed" area under the curve

ex: $\int_{-1}^{3}(x^3+1)\,dx = F(3)-F(-1)$

$F(x)=\frac{x^4}{4}+x$

$= \left[\frac{(3)^4}{4}+3\right]-\left[\frac{(-1)^4}{4}+(-1)\right]$

$= \frac{81}{4}+3-\frac{1}{4}+1 = \frac{80}{4}+4 = \boxed{24}$

ex: $\int_0^\pi \cos x\,dx = F(\pi)-F(0)$

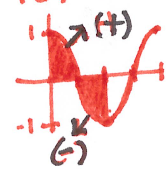
$F(x)=\sin x$

$= \sin\pi - \sin 0$

$= 0-0 \boxed{=0}$

*So there is no area under the curve from $[0,\pi]$ of $f(x)=\cos x$?

The (+) area cancels out with the (−) area!



* Finding **TOTAL** Area: If you can use a calculator: use the absolute value function!

No Calculator: Seperate the function into the positive areas & the negative areas. Then add $\underbrace{\text{the (+) ones}}_{\text{Above the x-axis}}$ & subtract $\underbrace{\text{the (−) areas}}_{\text{Below the x-axis}}$.

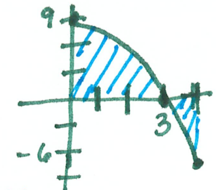
ex: $\int_0^{\pi} \cos x\,dx$ Find $\underline{\text{TOTAL}}$ area.

$\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx$ $\underline{\underline{\text{OR}}}$ $\int_0^{\pi} |\cos x|\,dx$ use the calculator! $\boxed{=2}$

$(\sin \pi/2 - \sin 0) - (\sin \pi - \sin \pi/2)$

$(1-0) - (0-1) = 1-(-1) \boxed{=2}$

ex: Find the TOTAL area between the graph & the x-axis from $[0,4]$ for $y=9-x^2$.



$\int_0^3 (9-x^2)dx - \int_3^4 (9-x^2)dx$

$= 9x - \frac{x^3}{3}\Big|_0^3 + \left[-9x + \frac{x^3}{3}\Big|_3^4\right]$

$\left(27-\frac{27}{3}\right) - 0 \quad \left(-36+\frac{64}{3}\right) + \left(+27 - \frac{27}{3}\right)$

$18 - 18 + \frac{64}{3} = \boxed{\frac{64}{3} \text{ or } 21.\overline{3}}$

ex: $\int_0^5 \left(\frac{9-x^2}{3x-9}\right)dx$

$\frac{(\cancel{3-x})(3+x)}{-3(\cancel{-x+3})} = \frac{3+x}{-3}$

*Hole @ $x=3$!

a) Can you use FTC, Part 2?

No! Discon't @ $x=3$.

b) Does the $\underline{\underline{\text{integral}}}$ have a value?

(Area under the curve) → yes!

$\int_0^5 \left(\frac{3+x}{-3}\right)dx = \int_0^5 \left(-1-\frac{x}{3}\right)dx = -x - \frac{x^2}{6}\Big|_0^5$

$= \left(-5-\frac{25}{6}\right) - \left(0-\frac{0}{6}\right) \boxed{= \frac{-55}{6}}$

ex: If $y = \int_1^x \frac{1}{t^2}\,dt$, Find $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{d}{dx}\left[\int_1^x \frac{1}{t^2}\,dt\right] = \frac{1}{x^2}(1) = \boxed{\frac{1}{x^2}}$

ex: If $y = \int_{3x^2}^{5x} \ln(2+p^2)\,dp$, Find $\frac{dy}{dx}$.

flip this & make it (−)!

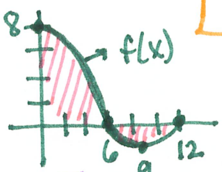
$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{3x^2}^{5x} \ln(2+p^2)dp\right] = \frac{d}{dx}\left[\int_{3x^2}^{0} \ln(2+p^2)dp + \int_0^{5x} \ln(2+p^2)dp\right]$

$= -\ln\left[2+(3x^2)^2\right]\cdot(6x) + \ln\left[2+(5x)^2\right]\cdot 5$

$\boxed{= -6x\ln(2+9x^4) + 5\ln(2+25x^2)}$

#57

$H(x) = \int_0^x f(t)\,dt$



a) $H(0) = \int_0^0 f(t)\,dt \boxed{=0}$

b) H increasing? → $H'>0$ $\quad H' = \frac{d}{dx}\int_0^x f(t)\,dt$

so... $\boxed{[0,6)}$ $\quad H' = f(x)$ → graph!

c) H ccup? → $H''>0$ $\quad H'' = f'(x)$ → slope of graph

so... $\boxed{(9,12)}$

d) $H(12) = \int_0^{12} f(t)\,dt$ → Area under the curve

$H(12)$ is (+): more (+) than (−).

e) Max of H? → $H'=0$ & change from (+) to (−).

$\therefore \boxed{x=6}$ $\quad x = 6, 12$

+ −

0 6 12

max

f) min of H? → H'

min min

$H(0)=0$

$H(12)>0$ $\quad \therefore$ min @ $\boxed{x=0}$