Ch. 10 Review: 2-62 e.o.e, 56, 60, 64

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n\cdot 3^n}$

$$\lim_{n\to\infty} \frac{|x+4|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\cdot 3^n}{|x+4|^n} = \left(\cancel{\lim_{n\to\infty} \frac{n}{n+1}}^{\,1}\right)\cdot \frac{|x+4|}{3} = \frac{|x+4|}{3} < 1$$

$|x+4| < 3 \rightarrow \boxed{R=3}$

$-7 < x < -1$

$x=-7$: $\frac{(-7+4)^n}{n\cdot 3^n} = \frac{(-3)^n}{n\cdot 3^n} = \left(\frac{-3}{3}\right)^n \cdot \frac{1}{n} = (-1)^n \cdot \frac{1}{n} \rightarrow$ Conv. Cond.

$x=-1$: $\frac{(-1+4)^n}{n\cdot 3^n} = \frac{\cancel{3^n}}{n\cdot \cancel{3^n}} = \frac{1}{n} \rightarrow$ Div.

IOC: $\boxed{[-7,-1)}$, Abs: $\boxed{(-7,-1)}$, Cond: $\boxed{x=-7}$

6. $\displaystyle\sum_{n=0}^{\infty} (n+1)x^{3n}$

$$\lim_{n\to\infty} \frac{(n+2)|x|^{3n+3}}{(n+1)|x|^{3n}} = \left(\cancel{\lim_{n\to\infty} \frac{n+2}{n+1}}^{\,1}\right)|x|^3 = |x|^3 < 1$$

$|x| < 1 \rightarrow \boxed{R=1}$

$-1 < x < 1$

$x=-1$: $(n+1)(-1)^{3n} \rightarrow$ Div bc $\lim_{n\to\infty} n+1 = \infty + 1 = \infty \neq 0$

$x=1$: $(n+1)(1)^{3n} \rightarrow$ Div bc $\lim_{n\to\infty} n+1 = \infty + 1 = \infty \neq 0$

IOC: $\boxed{(-1,1)}$, Abs: $\boxed{(-1,1)}$, Cond: $\boxed{\text{None}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(ex)^n}{n^e}$

$$\lim_{n\to\infty} \frac{|ex|^{n+1}}{(n+1)^e} \cdot \frac{n^e}{|ex|^n} = \left(\cancel{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^e}^{\,1^e = 1}\right)\cdot |ex| = |ex| < 1$$

$|x| < 1/e \rightarrow \boxed{R = 1/e}$

$-1/e < x < 1/e$

$x=-1/e$: $\frac{(\cancel{e}(-1/\cancel{e}))^n}{n^e} = (-1)^n \cdot \frac{1}{n^e} \leftarrow p=e>1 \rightarrow$ Con. Abs.

$x=1/e$: $(\cancel{e}\cdot 1/\cancel{e})^n \cdot \frac{1}{n^e} = (1)^n \cdot \frac{1}{n^e} \leftarrow p=e>1 \rightarrow$ Con. Abs.

IOC: $[-1/e, 1/e]$
Abs: $[-1/e, 1/e]$
Cond: None

14. $\displaystyle\sum_{n=2}^{\infty} \frac{(10x)^n}{\ln n}$

$$\lim_{n\to\infty} \frac{|10x|^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{|10x|^n} = \left(\lim_{n\to\infty} \cancel{\frac{\ln n}{\ln(n+1)}}\right) \cdot |10x| = |10x| < 1$$

$$|x| < 1/10 \to \boxed{R = 1/10}$$

$-1/10 < x < 1/10$

$x = -1/10$: $\dfrac{(\cancel{10} \cdot \frac{-1}{\cancel{10}})^n}{\ln(n)} = (-1)^n \cdot \dfrac{1}{\ln(n)} \to$ Con. Cond.

$x = 1/10$: $\dfrac{(\cancel{10} \cdot 1/\cancel{10})^n}{\ln(n)} = \cancel{(1)^n} \cdot \dfrac{1}{\ln(n)} \to$ Div. bc $\dfrac{1}{\ln(n)} \geq \dfrac{1}{n}$, which Div.

IOC: $\boxed{[-1/10, 1/10)}$, Abs: $\boxed{(-1/10, 1/10)}$, Cond: $\boxed{x = -1/10}$

18. $\dfrac{2}{3} - \dfrac{4}{18} + \dfrac{8}{81} + \ldots + (-1)^{n-1} \dfrac{1}{n} \cdot (2/3)^n = (-1)^{n-1} \dfrac{x^n}{n}$ at $x = 2/3$

Find which series has general term $= (-1)^{n-1} \dfrac{x^n}{n}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \ldots \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!}$$

We have $\ln(1+x)$ at $x = 2/3$, so $\ln(1 + 2/3) = \boxed{\ln(5/3)}$

22. $\dfrac{1}{1+x^2}$ $\quad a_0 = 1$, $r = -x^2$ $\quad = 1 - x^2 + x^4 - x^6 + \ldots$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \frac{(-1)^n x^{2n+1}}{2n+1} = \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$$

Given general term $= (-1)^{n-1} \dfrac{1}{(2n-1)(\sqrt{3})^{2n-1}} = \dfrac{(-1)^{n-1} (1/\sqrt{3})^{2n-1}}{2n-1}$

We have $\tan^{-1} x$ at $x = 1/\sqrt{3}$, so $\tan^{-1}(1/\sqrt{3}) = \boxed{\pi/6}$

26. $\frac{4x}{1-x}$ $\quad a_0 = 4x,\ r = x$

$$4x + 4x^2 + 4x^3 + 4x^4 + \ldots + 4x(x)^n = 4x^{n+1}$$

30. $\frac{1}{2}(e^x + e^{-x})$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \ldots$$

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}$$

34. $\tan^{-1} 3x$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \quad \begin{array}{l} a_0 = 1 \\ r = -x^2 \end{array} \quad 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$$

$$\tan^{-1} 3x = 3x - \frac{3^3x^3}{3} + \frac{3^5x^5}{5} - \frac{3^7x^7}{7} + \frac{3^9x^9}{9} - \ldots + \frac{(-1)^n(3x)^{2n+1}}{2n+1}$$

38.

$$\begin{array}{llll} f(x) = x^3 - 2x^2 + 5 & \text{at } x = -1 & = -1 - 2 + 5 & = 2 \\ f'(x) = 3x^2 - 4x & & = 3 + 4 & = 7 \\ f''(x) = 6x - 4 & & = -6 - 4 & = -10 \\ f'''(x) = 6 & & = 6 & \end{array}$$

$$2 + \frac{7}{1!}(x+1) - \frac{10}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3 = 2 + 7(x+1) - 5(x+1)^2 + (x+1)^3$$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{1/2}} \leftarrow p = 1/2 < 1$, so doesn't con. absolutely

- All positive, then signs alternate
- Value of terms decreasing
- $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{\infty}} = 0$

Con. $\boxed{\text{Conditionally}}$ by Alternating Series Test

46. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \leq \sum_{n=2}^{\infty} \frac{1}{n \cdot n^2} = \frac{1}{n^3} \leftarrow p = 3 > 1 \rightarrow$ Con. absolutely

Con. $\boxed{\text{Absolutely}}$ by Direct Comparison Test

50. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \cdot n \cdot n}} = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \leftarrow p = 3/2 > 1 \rightarrow$ Con. absolutely

Con. $\boxed{\text{Absolutely}}$ by Direct Comparison Test

54. $\displaystyle\sum_{n=2}^{\infty} \frac{-2}{n(n+1)} = \frac{2}{n+1} - \frac{2}{n}$

$\displaystyle\frac{-2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \rightarrow A(n+1) + B(n) = -2$

$n = -1: \quad -B = -2 \rightarrow B = 2$

$n = 0: \quad A = -2$

$$\underbrace{\cancel{\frac{2}{3}} - \frac{2}{2}}_{n=2} + \underbrace{\cancel{\frac{2}{4}} - \cancel{\frac{2}{3}}}_{n=3} + \underbrace{\cancel{\frac{2}{5}} - \cancel{\frac{2}{4}}}_{n=4} + \underbrace{\cancel{\frac{2}{6}} - \cancel{\frac{2}{5}}}_{n=5} + \cdots = \frac{-2}{2} = \boxed{-1}$$

58. $f(x) = \dfrac{1}{1-2x}$ $\quad a_0 = 1,\ r = 2x \quad \sum_{n=0}^{\infty} (2x)^n$

a) $\boxed{1 + 2x + 4x^2 + 8x^3 + \dots + (2x)^n}$

b) $r = 2x$ and $|r| < 1$, so $|2x| < 1 \to |x| < 1/2 \to \boxed{-1/2 < x < 1/2}$

$x = -1/2: \ (2 \cdot \frac{-1}{2})^n = (-1)^n = -1 + 1 - 1 + 1 - \dots \to$ Div.

$x = 1/2: \ (2 \cdot \frac{1}{2})^n = 1^n = 1 + 1 + 1 + 1 + \dots \to$ Div.

c) $f(-1/4) = \dfrac{1}{1-2(-1/4)} = \dfrac{1}{1+1/2} = \dfrac{1}{3/2} = \dfrac{2}{3} = 0.66\overline{6}$

1% of $f(-1/4) = 0.01 \times 0.66\overline{6} = 0.0066\overline{6}$

# of terms	Value	Error
1	1	$1/3 \approx 0.333$
2	$1 + 2(-1/4) = 0.5$	$1/6 \approx 0.167$
3	$1 + 2(-1/4) + 4(-1/4)^2 = 3/4$	$1/12 \approx 0.083$
4	$3/4 + 8(-1/4)^3 = 0.625$	0.0417
5	$0.625 + 16(-1/4)^4 = 0.6875$	0.0208
6	$0.6875 + 32(-1/4)^5 = 0.65625$	0.0104
$\boxed{7}$	$0.65625 + 64(-1/4)^6 = 0.671875$	$0.00521 < 0.00666$

62. $f(x) = \dfrac{x^2}{1+x}$ $\quad a_0 = x^2,\ r = -x$

a) $\boxed{x^2 - x^3 + x^4 - x^5 + \dots + (x^2)(-x)^n = (-1)^n x^{n+2}}$

b) At $x = 1$: $1 - 1 + 1 - 1 + 1 - 1$

The value of the partial sums alternate between 1 & 0, so the sum is not approaching a single finite value and the series does $\boxed{\text{not}}$ converge at $x = 1$.

56. $P_4(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^7$

a) $f(4) = \boxed{7}$

$\frac{f'''(4)}{3!} = -2 \rightarrow \frac{f'''(4)}{6} = -2 \rightarrow f'''(4) = \boxed{-12}$

b) $f'(x) = \boxed{-3 + 10(x-4) - 6(x-4)^2}$

$f'(4.3) = -3 + 10(4.3-4) - 6(4.3-4)^2 = \boxed{-0.54}$

c) $g(x) = \boxed{7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4}$

d) $\boxed{\text{No}}$, we would need an infinite series to determine an exact value, and that series would need to converge at $x=3$.

60. $f(x) = \frac{1}{x-2} = \frac{-1}{2-x} = (x-2)^{-1}$ at $x=3$ $= 1$

$f'(x) = -(x-2)^{-2} = \frac{-1}{(x-2)^2}$ $= -1$

$f''(x) = 2(x-2)^{-3} = \frac{2}{(x-2)^3}$ $= 2$

$f'''(x) = -6(x-2)^{-4} = \frac{-6}{(x-2)^4}$ $= -6$

a) $T_3(x) = 1 - \frac{1}{1!}(x-3) + \frac{2}{2!}(x-3)^2 - \frac{6}{3!}(x-3)^3 = \boxed{1 - (x-3) + (x-3)^2 - (x-3)^3 + \dots + (-1)^n(x-3)^n}$

b) $\ln|x-2| = \int \frac{1}{x-2}\,dx = \boxed{(x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 - \frac{1}{4}(x-3)^4 + \dots + \frac{(-1)^n(x-3)^{n+1}}{n+1}}$

c) $\ln 3/2 = \ln(3.5-2) \rightarrow x = 3.5$

$\ln 3/2 = 0.405465$ within $0.05 \rightarrow 0.355465$ to 0.455465

# of terms	Sum
1	0.5
2	$\boxed{0.375}$ ← within 0.05 of $\ln(3/2)$
3	0.417

64. $\int_0^1 x^2e^x\,dx$

a) Trapezoids

$$T=\frac{1}{2}\cdot\frac{1}{2}\left(0^2e^0+2\cdot 0.5^2e^{0.5}+1^2e^1\right)=\boxed{0.886}$$

b) $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

$x^2e^x = x^2+x^3+\frac{x^4}{2}+\ldots$

$$\int_0^1 \left(x^2+x^3+\tfrac{1}{2}x^4\right)dx = \left[\tfrac{1}{3}x^3+\tfrac{1}{4}x^4+\tfrac{1}{10}x^5\right]_0^1 = \frac{1}{3}+\frac{1}{4}+\frac{1}{10} = \boxed{0.683}$$

c) When a curve is concave up, the trapezoidal approximations lie above the curve. Therefore, the estimate is too large.

d) If all derivatives are positive, then each additional term used will produce a greater sum. Any approximation will be too small because not all of the positive terms will be used.

e) $\int_0^1 x^2e^x\,dx$ by parts (or tabular)

$$\begin{array}{ccc} x^2 & & e^x \\ 2x & + & e^x \\ 2 & - & e^x \\ 0 & + & e^x \end{array}$$

$$\left[e^x(x^2-2x+2)\right]_0^1 = e(1-\cancel{2}+\cancel{2})-1(2) = \boxed{e-2} \approx 0.718$$