Ch. 10 Review: 1-15 odd, 17-39 odd, 41-53 odd, 57-65 e.o.o.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{|-x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|-x|^n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot |-x|^{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot |-x|^1$

$\displaystyle\lim_{n\to\infty} \frac{1}{n+1} \cdot |-x| = \frac{1}{\infty+1} \cdot |-x| = 0 \cdot |-x| = 0 < 1$ for all x

Always converges

$\boxed{\text{a) } \infty \quad \text{b) } (-\infty, \infty) \quad \text{c) } (-\infty, \infty) \quad \text{d) None}}$

3. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n (x-1)^n = \left(\frac{2}{3}(x-1)\right)^n$

$r = \frac{2}{3}(x-1) \to \left|\frac{2}{3}(x-1)\right| < 1 \to |x-1| < 3/2 \to$ cen: $x=1$, R: $3/2$

IOC: $(-1/2, 5/2)$

$x = \frac{-1}{2}$: $\left(\frac{2}{3}\left(\frac{-3}{2}\right)\right)^n = (-1)^n \to r = -1 \to$ Div.

$x = \frac{5}{2}$: $\left(\frac{2}{3}\left(\frac{3}{2}\right)\right)^n = (1)^n \to r = 1 \to$ Div.

$\boxed{\text{a) } 3/2 \quad \text{b) } (-1/2, 5/2) \quad \text{c) } (-1/2, 5/2) \quad \text{d) None}}$

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(3x-1)^n}{n^2}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{|3x-1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|3x-1|^n} = \lim_{n\to\infty} \cancel{\frac{n^2}{n^2+2n+1}}^{\,1} \cdot |3x-1|^{\cancel{n}+1-\cancel{n}}$

$r = 1 \cdot |3x-1|^1 = |3x-1| < 1 \to |x - 1/3| < 1/3 \to$ cen: $x = 1/3$, R: $1/3$

IOC: $(0, 2/3)$

$x=0$: $(-1)^{n-1}(-1)^n \cdot \frac{1}{n^2} = (-1)^{2n-1} \cdot \frac{1}{n^2} \to p = 2 > 1 \to$ Con. Abs. by p-series

$x = 2/3$: $(-1)^{n-1} \cdot \cancel{1^n} \cdot \frac{1}{n^2} = (-1)^{n-1} \cdot \frac{1}{n^2} \to p = 2 > 1 \to$ Con. Abs. by p-series

$\boxed{\text{a) } 1/3 \quad \text{b) } [0, 2/3] \quad \text{c) } [0, 2/3] \quad \text{d) None}}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{2^n(2n+1)}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{(n+2)|2x+1|^{n+1}}{2^{n+1}(2n+3)} \cdot \frac{2^n(2n+1)}{(n+1)|2x+1|^n}$

$\displaystyle\lim_{n\to\infty} \frac{2n^2+\cancel{5}^{1}n+2}{\cancel{2n^2}+5n+3} \cdot 2^{\cancel{n}-\cancel{n}-1} \cdot |2x+1|^{\cancel{n}+1-\cancel{n}} = 1 \cdot 2^{-1} \cdot |2x+1|^1 = \frac{|2x+1|}{2} = r$

$\frac{|2x+1|}{2} < 1 \to |2x+1| < 2 \to |x+1/2| < 1 \to$ cen : $x = -1/2$, R : 1

IOC : $(-3/2, 1/2)$

$x = -3/2$: $\frac{(-2)^n(n+1)}{2^n(2n+1)} = \left(\frac{-2}{2}\right)^n \cdot \frac{n+1}{2n+1} = (-1)^n \cdot \frac{n+1}{2n+1} \to$ Div. bc $\displaystyle\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} \neq 0$

$x = 1/2$: $\frac{\cancel{2^n}(n+1)}{\cancel{2^n}(2n+1)} = \frac{n+1}{2n+1} \to$ Div. bc $\displaystyle\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} \neq 0$

$\boxed{\text{a) } 1 \quad \text{b) } (-3/2, 1/2) \quad \text{c) } (-3/2, 1/2) \quad \text{d) None}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x|^n} = \lim_{n\to\infty} \cancel{\frac{\sqrt{n}}{\sqrt{n+1}}}^{1} \cdot |x|^{\cancel{n}+1-\cancel{n}} = 1 \cdot |x|^1 = |x| = r$

$|x| < 1 \to$ Cen: $x = 0$, R : 1 , IOC : $(-1, 1)$

$x = -1$: $(-1)^n \cdot \frac{1}{n^{1/2}}$ Con. Cond. by Alt. Series Test

$x = 1$: $\cancel{(1)^n} \cdot \frac{1}{n^{1/2}} \to$ Div. by p-series w/ $p = 1/2 < 1$

$\boxed{\text{a) } 1 \quad \text{b) } [-1, 1) \quad \text{c) } (-1, 1) \quad \text{d) } x = -1}$

11. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

Ratio Test: $\lim_{n\to\infty} \frac{(n+2)|x|^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)|x|^{2n-1}}$

$\lim_{n\to\infty} \frac{n+2}{n+1} \cdot |x|^{\cancel{2n}+1-\cancel{2n}+1} \cdot 3^{\cancel{n}-\cancel{n}-1} = 1 \cdot |x|^2 \cdot 3^{-1} = \frac{|x|^2}{3} = r$

$\frac{|x|^2}{3} < 1 \rightarrow |x|^2 < 3 \rightarrow |x| < \sqrt{3} \rightarrow$ Cen: $x=0$, R: $\sqrt{3}$, IOC: $(-\sqrt{3}, \sqrt{3})$

$x=-\sqrt{3}$: $\frac{(n+1)(-\sqrt{3})^{2n-1}}{3^n} \rightarrow \lim_{n\to\infty} (n+1) \cdot \frac{(-\sqrt{3})^{2n-1}}{3^n} = (\infty+1) \cdot \frac{(-\sqrt{3})^{2n-1}}{3^n} = \infty$

Div. at $x=-\sqrt{3}$ (same at $x=\sqrt{3}$) by the n^{th} Term Test.

| a) $\sqrt{3}$ b) $(-\sqrt{3}, \sqrt{3})$ c) $(-\sqrt{3}, \sqrt{3})$ d) None |

13. $\sum_{n=1}^{\infty} \frac{n!\,x^{2n}}{2^n}$

Ratio Test: $\lim_{n\to\infty} \frac{(n+1)!\,|x|^{2n+2}}{2^{n+1}} \cdot \frac{2^n}{n!\,|x|^{2n}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot |x|^{\cancel{2n}+2-\cancel{2n}} \cdot 2^{\cancel{n}-\cancel{n}-1}$

$\lim_{n\to\infty} (n+1) \cdot |x|^2 \cdot 2^{-1} = (\infty+1) \cdot \frac{|x|^2}{2} = \infty > 1$ for all x

Only converges at $x=0$.

| a) 0 b) $x=0$ c) $x=0$ d) None |

5. $\sum_{n=1}^{\infty} (n+1)!\,x^n$

Ratio Test: $\lim_{n\to\infty} \frac{(n+2)!\,|x|^{n+1}}{(n+1)!\,|x|^n} = \lim_{n\to\infty} (n+2) \cdot |x|^{\cancel{n}+1-\cancel{n}} = (\infty+2) \cdot |x| = \infty > 1$ for all x

Only converges at $x=0$.

| a) 0 b) $x=0$ c) $x=0$ d) None |

17. $1-\frac{1}{4}+\frac{1}{16}-\dots = \left(\frac{1}{4}\right)^0-\left(\frac{1}{4}\right)^1+\left(\frac{1}{4}\right)^2-\dots (-1)^n\left(\frac{1}{4}\right)^n$

In general, $1-x+x^2-\dots$, so $a_0=1$ and $r=-x$

$S=\frac{a_0}{1-r}$, so $f(x)=\frac{1}{1-(-x)}=\boxed{\frac{1}{1+x}}$ at $\boxed{x=1/4}$

$S=\frac{1}{1-(\frac{1}{4})}=\frac{1}{1+\frac{1}{4}}=\frac{1}{5/4}=\boxed{\frac{4}{5}}$

19. $\pi-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\dots$

In general, $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$, so $f(x)=\boxed{\sin x}$ at $\boxed{x=\pi}$

Sum: $f(\pi)=\sin\pi=\boxed{0}$

21. $1+\ln 2+\frac{(\ln 2)^2}{2!}+\dots+\frac{(\ln 2)^n}{n!}$

In general, $1+x+\frac{x^2}{2!}+\dots\frac{x^n}{n!}$, so $f(x)=\boxed{e^x}$ at $\boxed{x=\ln 2}$

Sum: $f(\ln 2)=\cancel{e}^{\cancel{\ln} 2}=\boxed{2}$

23. $\frac{1}{1-6x}\rightarrow \begin{array}{l} a_0=1 \\ r=6x \end{array}$

$\boxed{1+6x+36x^2+216x^3+\dots+(6x)^n}$

25. x^9-2x^2+1

Just change order from smallest to largest degree: $\boxed{1-2x^2+x^9}$

27. $\sin(\pi x)$

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\sin(\pi x)=\boxed{\pi x-\frac{(\pi x)^3}{3!}+\frac{(\pi x)^5}{5!}-\frac{(\pi x)^7}{7!}+\dots+\frac{(-1)^n(\pi x)^{2n+1}}{(2n+1)!}}$

29. $-x+\sin x$

$-\cancel{x}+\cancel{x}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots=\boxed{\underbrace{-\frac{x^3}{3!}}_{n=1}+\underbrace{\frac{x^5}{5!}}_{n=2}\underbrace{-\frac{x^7}{7!}}_{n=3}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}}$

31. $\cos\sqrt{5x}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos\sqrt{5x} = 1 - \frac{(\sqrt{5x})^2}{2!} + \frac{(\sqrt{5x})^4}{4!} - \frac{(\sqrt{5x})^6}{6!} + \dots + \frac{(-1)^n(\sqrt{5x})^{2n}}{(2n)!}$$

$$\cos\sqrt{5x} = \boxed{1 - \frac{(5x)^1}{2!} + \frac{(5x)^2}{4!} - \frac{(5x)^3}{6!} + \dots + \frac{(-1)^n(5x)^n}{(2n)!}}$$

33. xe^{-x^2}

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots + \frac{(-x^2)^n}{n!}$$

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!}$$

$$xe^{-x^2} = \boxed{x - \frac{x^3}{1!} + \frac{x^5}{2!} - \frac{x^7}{3!} + \dots + \frac{(-1)^n x^{2n+1}}{n!}}$$

35. $\ln(1-2x)$

$$\frac{d}{dx}(\ln(1-2x)) = \frac{1}{1-2x} \cdot -2 = \frac{-2}{1-2x}$$

$$\frac{-2}{1-2x} \rightarrow \begin{matrix} a_0 = -2 \\ r = 2x \end{matrix} \rightarrow -2 - 4x - 8x^2 - 16x^3 - \dots - 2(2x)^n = -2^{n+1}x^n$$

$$\ln(1-2x) = \int(-2-4x-8x^2-16x^3-\dots)dx = \boxed{-2x - \frac{4x^2}{2} - \frac{8x^3}{3} - \frac{16x^4}{4} - \dots \frac{-2^{n+1}x^{n+1}}{n+1}}$$

37. $f(x) = (3-x)^{-1}$, $a = 2$

$f'(x) = -1(3-x)^{-2}(-1) = 1(3-x)^{-2}$

$f''(x) = -2(3-x)^{-3}(-1) = 2(3-x)^{-3}$

$f'''(x) = -6(3-x)^{-4}(-1) = 6(3-x)^{-4}$

$f(2) = 1$

$f'(2) = 1 \cdot 1 = 1$

$f''(2) = 2 \cdot 1 = 2$

$f'''(2) = 6 \cdot 1 = 6$

$$1 + \frac{1(x-2)^1}{1!} + \frac{2(x-2)^2}{2!} + \frac{6(x-2)^3}{3!} = \boxed{1 + (x-2) + (x-2)^2 + (x-2)^3}$$

General term: $\boxed{(x-2)^n}$

39. $f(x) = \frac{1}{x} = x^{-1}$, $a = 3$ $\quad f(3) = 1/3$

$f'(x) = -1x^{-2} = \frac{-1}{x^2}$ $\quad f'(3) = -1/9$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$ $\quad f''(3) = 2/27$

$f'''(x) = -6x^{-4} = \frac{-6}{x^4}$ $\quad f'''(3) = -6/81 = -2/27$

$$\frac{1}{3} + \frac{-1/9\,(x-3)^1}{1!} + \frac{2/27\,(x-3)^2}{2!} - \frac{2/27\,(x-3)^3}{3!}$$

$$\boxed{\frac{1}{3} - \frac{1}{9}(x-3) + \frac{1}{27}(x-3)^2 - \frac{1}{81}(x-3)^3 + \ldots + (-1)^n \cdot \frac{1}{3^{n+1}}(x-3)^n}$$

$n=0$ $\quad n=1$ $\quad n=2$ $\quad n=3$

41. $\sum_{n=1}^{\infty} \frac{-5}{n} = -5 \cdot \frac{1}{n^1} = -5\,(\text{Div.}) = \boxed{\text{Div. by p-series}}$

43. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3} < \frac{n}{n^3} = \frac{1}{n^2} \to p = 2 > 1 \to$ Con. Abs. by p-series

Since $\ln(n) < n$, then $\frac{\ln(n)}{n^3} < \frac{n}{n^3}$, and $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ also $\boxed{\text{converges absolutely by Direct Comparison}}$

45. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\ln(n+1)}$ $\quad a_n = \frac{1}{\ln(n+1)}$ $\quad \frac{1}{\ln(n)} \approx \frac{1}{n} = b_n$

Limit Comp. Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\ln(n+1)} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{\ln(n+1)} = \frac{\infty}{\infty}$$

L'Hopital's Rule:

$$\lim_{n\to\infty} \frac{1}{\frac{1}{n+1}} = \lim_{n\to\infty} n+1 = \infty + 1 = \infty \to a_n \text{ grows faster than } b_n$$

$b_n = \frac{1}{n}$, which diverges (by p-series w/ $n=1$ or integral test).

a_n also diverges by Limit Comp Test, so the series does not converge absolutely.

Check for conditional convergence with Alt. Series Test:

1) Positive terms, then alt. signs applied ✓
2) Value of terms decreasing ✓
3) $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = \frac{1}{\ln(\infty+1)} = \frac{1}{\ln\infty} = \frac{1}{\infty} = 0$ ✓

The series $\boxed{\text{converges conditionally by the Alternating Series Test}}$.

47. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

Ratio Test (good for factorials & exponentials)

$\lim_{n\to\infty} \frac{(-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-3)^n} = \lim_{n\to\infty} (-3)^{\cancel{n}+1-\cancel{n}} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} \cdot -3 = \frac{1}{\infty+1}(-3) = 0$

$0<1$ always, so the series $\boxed{\text{converges absolutely by Ratio Test}}$

49. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2+1}{2n^2+n-1}$

$\lim_{n\to\infty} \frac{n^2+1}{2n^2+n-1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{2\cancel{n^2}} = \frac{1}{2} \neq 0 \to$ $\boxed{\text{Diverges by } n^{th} \text{ term test}}$

51. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ $b_n = \frac{1}{n\sqrt{n^2}} = \frac{1}{n \cdot n} = \frac{1}{n^2}$

Limit Comparison Test:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n\sqrt{n^2-1}} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} = 1 \to$ same for a_n & b_n

$b_n = \frac{1}{n^2} \to p = 2 > 1 \to$ Converges Absolutely by p-series

a_n also $\boxed{\text{converges absolutely by Limit Comparison Test}}$

53. $\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)} = \frac{A}{2n-3} + \frac{B}{2n-1}$

$A(2n-1) + B(2n-3) = 1$

$n = 1/2: \quad -2B = 1 \to B = -1/2$

$n = 3/2: \quad 2A = 1 \to A = 1/2$

$\sum_{n=3}^{\infty} \left(\frac{1/2}{2n-3} - \frac{1/2}{2n-1} \right)$

$\underbrace{\frac{1/2}{3}\left(-\cancel{\frac{1/2}{5}}\right.}_{n=3} + \underbrace{\cancel{\frac{1/2}{5}}\left(-\cancel{\frac{1/2}{7}}\right.}_{n=4} + \underbrace{\cancel{\frac{1/2}{7}}\left(-\cancel{\frac{1/2}{9}}\right.}_{n=5} + \ldots = \frac{1/2}{3} = \frac{1}{2} \cdot \frac{1}{3} = \boxed{\frac{1}{6}}$

57. $f(x) = 5\sin(\frac{1}{2}x)$ at $x = 0$

a) $f'(x) = \frac{5}{2}\cos(\frac{1}{2}x)$ $\quad f(0) = 0$

$f'(0) = \frac{5}{2} \cdot 1 = \frac{5}{2}$

$f''(x) = -\frac{5}{4}\sin(\frac{1}{2}x)$ $\quad f''(0) = 0$

$f^3(x) = -\frac{5}{8}\cos(\frac{1}{2}x)$ $\quad f^3(0) = \frac{-5}{8} \cdot 1 = \frac{-5}{8}$

$f^4(x) = +\frac{5}{16}\sin(\frac{1}{2}x)$ $\quad f^4(0) = 0$

$f^5(x) = \frac{5}{32}\cos(\frac{1}{2}x)$ $\quad f^5(0) = \frac{5}{32} \cdot 1 = \frac{5}{32}$

$$\frac{5/2\, x^1}{1!} - \frac{5/8\, x^3}{3!} + \frac{5/32\, x^5}{5!} = \boxed{\frac{5x^1}{2^1 \cdot 1!} - \frac{5x^3}{2^3 \cdot 3!} + \frac{5x^5}{2^5 \cdot 5!} - \ldots + \frac{(-1)^n 5 \cdot x^{2n+1}}{2^{2n+1}(2n+1)!}}$$

b) $$\sum_{n=0}^{\infty} \frac{(-1)^n 5 \cdot x^{2n+1}}{2^{2n+1}(2n+1)!}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{\cancel{5} \cdot |x|^{2n+3}}{2^{2n+3}(2n+3)!} \cdot \frac{2^{2n+1}(2n+1)!}{\cancel{5} \cdot |x|^{2n+1}} = \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} \cdot |x|^{\cancel{2n}+3-\cancel{2n}-1} \cdot 2^{\cancel{2n}+1-\cancel{2n}}$$

$$\lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} \cdot |x|^2 \cdot 2^{-2} = \frac{1}{\infty} \cdot \frac{|x|^2}{2^2} = 0 \cdot \frac{|x|^2}{2^2} = 0 < 1 \text{ always}$$

$\boxed{\text{Converges for all } x}$

c) One term:

$$\text{Error} = \left|5\sin(\tfrac{1}{2}x) - \tfrac{5}{2}x\right| = 0.793 \text{ at } x = \pm 2$$

Two terms:

$$\text{Error} = \left|5\sin(\tfrac{1}{2}x) - (\tfrac{5}{2}x - \tfrac{5}{48}x^3)\right| = 0.0407 < 0.1 \checkmark \rightarrow \boxed{\text{Two terms}}$$

61. $f(x) = e^{-2x^2}$

a) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

$$e^{-2x^2} = 1 - 2x^2 + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \ldots + \frac{(-2x^2)^n}{n!}$$

$$e^{-2x^2} = \boxed{1 - \frac{2^1x^2}{1!} + \frac{2^2x^4}{2!} - \frac{2^3x^6}{3!} + \ldots + \frac{(-1)^n 2^n x^{2n}}{n!}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n}}{n!}$$

b) Ratio Test:

$$\lim_{n\to\infty} \frac{2^{n+1}|x|^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^n \cdot |x|^{2n}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot 2^{\cancel{n}+1-\cancel{n}} \cdot |x|^{\cancel{2n}+2-\cancel{2n}}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot 2 \cdot |x|^2 = \frac{1}{\infty+1} \cdot 2|x|^2 = 0 \cdot 2|x|^2 = 0 < 1 \text{ always}$$

$\boxed{\text{Converges for all } x}$

61. c) Error $= \left|e^{-2x^2} - \left(1 - 2x^2 + 2x^4 - \frac{4}{3}x^6\right)\right| \leq 0.00976 < 0.02$ ✓

for $-0.6 \leq x \leq 0.6$

65. a) $A = a_n(1.08)^n \rightarrow 1000 = a_n(1.08)^n \rightarrow a_n = \dfrac{1000}{(1.08)^n} = \boxed{1000(1.08)^{-n}}$

b) n represents the year so n=1 is year 1, n=2 is year 2, etc.

$$\underset{\text{year 1}}{\underset{\uparrow}{1000(1.08)^{-1}}} + \underset{\text{year 2}}{\underset{\uparrow}{1000(1.08)^{-2}}} + \underset{\text{year 3}}{\underset{\uparrow}{1000(1.08)^{-3}}} + \ldots + \underset{\text{year } n}{\underset{\uparrow}{1000(1.08)^{-n}}}$$

$$\boxed{\sum_{n=1}^{\infty} 1000(1.08)^{-n}}$$

c) $a_1 = 1000(1.08)^{-1} = \dfrac{1000}{1.08}$

$r = \dfrac{1}{1.08} = 0.926 < 1 \rightarrow$ Converges by Geometric Series

$$S = \frac{a_1}{1-r} = \frac{1000/1.08}{1 - \frac{1}{1.08}} = \frac{\frac{1000}{1.08}}{\frac{1.08}{1.08} - \frac{1}{1.08}} = \frac{\frac{1000}{1.08}}{\frac{0.08}{1.08}} = \frac{1000}{0.08} = \boxed{12{,}500}$$

This means that initially investing \$12,500 in the fund would be enough to completely fund the perpetuity indefinitely. You wouldn't need to invest any more to continue donating \$1,000 per year forever.