Ch. 4 Review: 1–57 e.o.o., 59–83 odd

1. $y = e^{3x-7}$

$y' = e^{3x-7} \cdot 3 = \boxed{3e^{3x-7}}$

5. $s = \cos(1-2t)$

$s' = -\sin(1-2t) \cdot -2 = \boxed{2\sin(1-2t)}$

9. $r = \sec(1+3\theta)$

$r' = \sec(1+3\theta) \cdot \tan(1+3\theta) \cdot 3 = \boxed{3\sec(1+3\theta)\tan(1+3\theta)}$

13. $y = \ln(1+e^x)$

$y' = \frac{1}{1+e^x} \cdot e^x = \boxed{\frac{e^x}{1+e^x}}$

17. $r = \ln(\cos^{-1}x)$

$r' = \frac{1}{\cos^{-1}x} \cdot \frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-1}{\cos^{-1}x\sqrt{1-x^2}}}$

21. $y = x^{\ln x}$

$\ln y = \ln x \cdot \ln x$

$\frac{1}{y}\frac{dy}{dx} = \ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{x}$

$\frac{dy}{dx} = y\left(\frac{2\ln x}{x}\right) = \frac{x^{\ln x} \cdot 2\ln x}{x} = \frac{2(\ln x)x^{\ln x}}{x} = \boxed{2(\ln x)x^{\ln x - 1}}$

25. $y = t \cdot \sec^{-1}t - \frac{1}{2}\ln t$

$y' = t \cdot \frac{1}{|t|\sqrt{t^2-1}} + \sec^{-1}t \cdot 1 - \frac{1}{2t} = \boxed{\frac{t}{|t|\sqrt{t^2-1}} + \sec^{-1}t - \frac{1}{2t}}$

29. $y = \csc^{-1}(\sec x)$

$y' = \frac{-1}{|\sec x|\sqrt{\sec^2 x - 1}} \cdot \sec x \cdot \tan x = \frac{-\sec x \tan x}{|\sec x|\sqrt{\tan^2 x}} = \frac{-\cancel{\sec x} \cdot \cancel{\tan x}}{|\cancel{\sec x}| \cdot \cancel{\tan x}} = \boxed{-1}$

$\sin^2 x + \cos^2 x = 1 \rightarrow \tan^2 x + 1 = \sec^2 x \rightarrow \sec^2 x - 1 = \tan^2 x$

33. $y = \left(\frac{1-x}{1+x^2}\right)^{1/2} = \sqrt{\frac{1-x}{1+x^2}}$

Inside $\geq 0 \rightarrow \frac{1-x}{1+x^2} \geq 0$

$1+x^2$ is always positive, so $1-x \geq 0 \rightarrow x \leq 1$

Endpoint at $x=1$, so $\boxed{x<1}$

37. $\sqrt{xy} = 1$

$\frac{1}{2}(xy)^{-1/2}\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$

$\frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx} + y\right) = 0$

$\frac{x\frac{dy}{dx}}{2\sqrt{xy}} + \frac{y}{2\sqrt{xy}} = 0 \rightarrow \frac{x\frac{dy}{dx}}{2\sqrt{xy}} = \frac{-y}{2\sqrt{xy}}$

$x\frac{dy}{dx} = -y$

$\frac{dy}{dx} = \boxed{\frac{-y}{x}}$

41. $y^3 + y = 2\cos x$

$3y^2\frac{dy}{dx} + 1\frac{dy}{dx} = -2\sin x$

$\frac{dy}{dx} = \frac{-2\sin x}{3y^2+1}$

$$\frac{d^2y}{dx^2} = \frac{(3y^2+1)(-2\cos x) - (-2\sin x)(6y)\frac{dy}{dx}}{(3y^2+1)^2} = \frac{\frac{(3y^2+1)(-2\cos x)}{1} + 2\sin x(6y)\cdot\frac{-2\sin x}{3y^2+1}}{(3y^2+1)^2}$$

$$= \frac{\frac{(3y^2+1)^2(-2\cos x)}{3y^2+1} - \frac{24y\sin^2 x}{3y^2+1}}{(3y^2+1)^2} = \frac{-2\cos x(3y^2+1)^2 - 24y\sin^2 x}{3y^2+1} \cdot \frac{1}{(3y^2+1)^2}$$

$$= \boxed{\frac{-2\cos x(3y^2+1)^2 - 24y\sin^2 x}{(3y^2+1)^3}}$$

45. $y = (x^2-2x)^{1/2}$

$y = \sqrt{3^2 - 2\cdot 3} = \sqrt{9-6} = \sqrt{3} \rightarrow (3, \sqrt{3})$

$y' = \frac{1}{2}(x^2-2x)^{-1/2}(2x-2) = \frac{x-1}{\sqrt{x^2-2x}}$ at $x=3 \rightarrow \frac{3-1}{\sqrt{3^2-2\cdot 3}} = \frac{2\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2\sqrt{3}}{3}$

a) Tangent: $y - \sqrt{3} = \frac{2\sqrt{3}}{3}(x-3)$

$y - \sqrt{3} = \frac{2\sqrt{3}}{3}x - 2\sqrt{3}$

$\boxed{y = \frac{2\sqrt{3}}{3}x - \sqrt{3}}$

b) Normal: $m = \frac{-\sqrt{3}}{2}$ at $(3, \sqrt{3})$

$y - \sqrt{3} = \frac{-\sqrt{3}}{2}(x-3)$

$y - \sqrt{3} = \frac{-\sqrt{3}}{2}x + \frac{3\sqrt{3}}{2}$

$\boxed{y = \frac{-\sqrt{3}}{2}x + \frac{5\sqrt{3}}{2}}$

49. $x = 2\sin t,\ y = 2\cos t,\ t = 3\pi/4$

$x = 2\sin\frac{3\pi}{4} = \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}} = \sqrt{2}$

$y = 2\cos\frac{3\pi}{4} = \cancel{2}\cdot\frac{-\sqrt{2}}{\cancel{2}} = -\sqrt{2}$

$\Big\rangle\ (\sqrt{2}, -\sqrt{2})$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\cancel{2}\sin t}{\cancel{2}\cos t} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$

$y + \sqrt{2} = 1(x - \sqrt{2})$

$\boxed{y = x - 2\sqrt{2}}$

53. $f(x) = \begin{cases} \sin ax + b\cos x, & x < 0 \quad \text{Left} \\ 5x + 3, & x \geq 0 \quad \text{Right} \end{cases}$

a) $5(0) + 3 = 0 + 3 = 3$ y value as $x \to 0^+$

$\sin 0 + b\cos 0 = 3$ to be continuous

$0 + b(1) = 3 \to \boxed{b = 3}$

b) Right: $y = 5x + 3 \to y' = 5 \to m = 5$ as $x \to 0^+$

Left: $y = \sin ax + 3\cos x$

$y' = a\cos ax - 3\sin x$

$y'(0) = a\cos 0 - 3\sin 0 = 5 \to a(1) - \cancel{3(0)} = 5 \to \boxed{a = 5}$

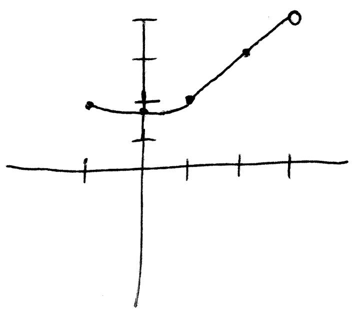
c) $f(x) = \begin{cases} \sin 5x + 4\cos x, & x < 0 \\ 5x + 3, & x \geq 0 \end{cases}$

$5(0) + 3 = 3 \to y$ value as $x \to 0^+$

$\sin 0 + 4\cos 0 = 0 + 4(1) = 4 \neq 5$ y value as $x \to 0^-$

If $b = 4$, the function is not continuous and therefore $\boxed{\text{not}}$ differentiable.

57. $f(x) = \begin{cases} \sqrt{x^2 + 3}, & -1 \leq x < 1 \\ x + 1, & 1 \leq x < 3 \end{cases}$



Check slope as $x \to 1^-$ and $x \to 1^+$

$y = x + 1 \to y' = 1 \to m = 1$ as $x \to 1^+$

$y = (x^2 + 3)^{1/2} \to y' = \cancel{\tfrac{1}{2}}(x^2 + 3)^{-1/2} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2 + 3}}$

$y'(1) = \frac{1}{\sqrt{1^2 + 3}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$ as $x \to 1^-$

$1 \neq \frac{1}{2} \to$ cusp at $x = 1 \to$ not differentiable

Not differentiable at endpoints: $x = -1,\ x = 3$

a) $(-1, 1) \cup (1, 3)$

b) $x = -1,\ x = 1$

c) $x = 3$

59. a) $y=(\sqrt{3-\sin x})^2 = 3-\sin x$

$y' = \boxed{-\cos x}$

b) $y = \ln(3e^{7x^2-13x+5}) = \ln 3 + \cancel{\ln e}^{7x^2-13x+5} = \ln 3 + 7x^2-13x+5 = 7x^2-13x+5+\ln 3$

$y' = \boxed{14x-13}$

c) $s = \cancel{\tan}(\cancel{\tan^{-1}}(t^2-3t)) = t^2-3t$

$s' = \boxed{2t-3}$

d) $s = \sqrt[3]{t^6} - 5(\cancel{\sin}(\cancel{\sin^{-1}} t))^6 = t^{6/3} - 5t^6 = t^2 - 5t^6$

$s' = \boxed{2t-30t^5}$

61. $f'(x) > 0 \rightarrow$ always increasing, $y=3x-2$ is tangent at $x=2$

a) Tan $=3$, Norm $= -1/3$ $\qquad \boxed{y-4 = -1/3(x-2)}$

$y = 3(2)-2 = 4 \rightarrow (2,4)$

b) $2 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 4$ $\qquad f^{-1}: (4,2)$ $\qquad f(f^{-1}(x)) = x$

$f'(f^{-1}(x)) \cdot (f^{-1}(x))' = 1$

$(f^{-1}(x))' = \dfrac{1}{f'(f^{-1}(x))}$ at $x=4$ $= \dfrac{1}{f'(f^{-1}(4))} = \dfrac{1}{f'(2)} = \dfrac{1}{3}$

$\boxed{y-2 = 1/3(x-4)}$

c) $y = \dfrac{f(x)}{x} \rightarrow y = \dfrac{f(2)}{2} = \dfrac{4}{2} = 2 \rightarrow (2,2)$ $\qquad \boxed{y-2 = \frac{1}{2}(x-2)}$

$y' = \dfrac{x\cdot f'(x) - f(x)\cdot 1}{x^2} \rightarrow y'(2) = \dfrac{2\cdot f'(2) - f(2)}{2^2} = \dfrac{2(3)-4}{4} = \dfrac{2}{4} = \dfrac{1}{2}$

63. $y = \dfrac{(x+2)^5(2x-3)^4}{(x+17)^2}$

$\ln y = \ln(x+2)^5 + \ln(2x-3)^4 - \ln(x+17)^2$

$\ln y = 5\ln(x+2) + 4\ln(2x-3) - 2\ln(x+17)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{5}{x+2} + \dfrac{4}{2x-3}\cdot\dfrac{2}{1} - \dfrac{2}{x+17}$

$\dfrac{dy}{dx} = y\left(\dfrac{5}{x+2} + \dfrac{8}{2x-3} - \dfrac{2}{x+17}\right) = \boxed{\dfrac{(x+2)^5(2x-3)^4}{(x+17)^2}\left(\dfrac{5}{x+2} + \dfrac{8}{2x-3} - \dfrac{2}{x+17}\right)}$

65. a) $f'(x) = x$

$f(x) = \boxed{\frac{1}{2}x^2}$

b) $f'(x) = f(x)$

$f(x) = \boxed{e^x}$

c) $f'(x) = -f(x)$

$f(x) = \boxed{e^{-x}}$

d) $f''(x) = f(x)$

$f(x) = \boxed{e^x}$

e) $f''(x) = -f(x)$

$f(x) = \boxed{\sin x}$

67. a) $\frac{f(2x)}{x-1} \rightarrow \frac{(x-1)\cdot f'(2x)\cdot 2 - f(2x)\cdot 1}{(x-1)^2}$ at $x=0 \rightarrow \frac{-2f'(0)-f(0)}{(-1)^2}$

$-2(-2) - -1 = 4+1 = \boxed{5}$

b) $f^2(x)\cdot g^3(x) \rightarrow f^2(x)\cdot 3g^2(x)\cdot g'(x) + g^3(x)\cdot 2f(x)\cdot f'(x)$ at $x=0$

$3(f(0))^2(g(0))^2 g'(0) + 2(g(0))^3 f(0) f'(0) = 3\cdot(-1)^2(-3)^2\cdot 4 + 2(-3)^3(-1)(-2) = 108 - 108 = \boxed{0}$

c) $g(f(x)) \rightarrow g'(f(x))\cdot f'(x)$ at $x=-1 \rightarrow g'(f(-1))\cdot f'(-1) = g'(0)\cdot f'(-1) = 4\cdot 2 = \boxed{8}$

d) $f(g(x)) \rightarrow f'(g(x))\cdot g'(x)$ at $x=-1 \rightarrow f'(g(-1))\cdot g'(-1) = f'(-1)\cdot g'(-1) = 2\cdot 1 = \boxed{2}$

e) $f(g(2x-1)) \rightarrow f'(g(2x-1))\cdot g'(2x-1)\cdot 2$ at $x=0 \rightarrow f'(g(-1))\cdot g'(-1)\cdot 2$

$f'(-1)\cdot g'(-1)\cdot 2 = 2\cdot 1\cdot 2 = \boxed{4}$

f) $g(x+f(x)) \rightarrow g'(x+f(x))\cdot(1+f'(x))$ at $x=0 \rightarrow g'(f(0))(1+f'(0))$

$g'(-1)(1+-2) = -g'(-1) = \boxed{-1}$

69. $\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$, $r = (\theta^2+7)^{1/3}$, $\theta^2 t + \theta = 1$ at $t=0$

$\theta^2\cdot 1 + t\cdot 2\theta\frac{d\theta}{dt} + 1\frac{d\theta}{dt} = 0$

$\frac{d\theta}{dt} = \frac{-\theta^2}{2\theta t+1}$

$\frac{dr}{dt} = \frac{1}{3}(\theta^2+7)^{-2/3}\cdot 2\theta\cdot\frac{d\theta}{dt}$

$\frac{dr}{dt} = \frac{2\theta}{3(\theta^2+7)^{2/3}}\cdot\frac{-\theta^2}{2\theta t+1}$

When $t=0$: $\theta^2\cancel{0} + \theta = 1$, then $\theta = 1$.

$\frac{dr}{dt} = \frac{-2\theta^3}{(6\theta t+3)(\theta^2+7)^{2/3}}$

$\frac{dr}{dt}$ at $t=0, \theta=1$: $\frac{-2\cdot 1^3}{(6\cdot 1\cdot 0+3)(1^2+7)^{2/3}} = \frac{-2}{3\cdot 8^{2/3}} = \frac{-2}{3\cdot(\sqrt[3]{8})^2} = \frac{-2}{3\cdot 2^2} = \frac{-2}{3\cdot 4} = \boxed{\frac{-1}{6}}$

71. $4x^2+8xy+y^2+3=0$

$8x+8x\frac{dy}{dx}+8y+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-8x-8y}{8x+2y}=\frac{-4x-4y}{4x+y}$

a) Horizontal when $-4x-4y=0$

$4x=-4y \rightarrow y=-x$

$4x^2+8x(-x)+(-x)^2+3=0$

$4x^2-8x^2+x^2+3=0$

$-3x^2=-3$

$x^2=1 \rightarrow x=\pm 1$ ↗ $x=1,\ y=-x=-1 \rightarrow \boxed{(1,-1)}$ ↘ $x=-1,\ y=-x=1 \rightarrow \boxed{(-1,1)}$

b) Vertical when $4x+y=0 \rightarrow y=-4x$

$4x^2+8x(-4x)+(-4x)^2+3=0$

$4x^2-32x^2+16x^2+3=0$

$-12x^2=-3 \rightarrow x^2=1/4 \rightarrow x=\pm 1/2$ ↗ $x=1/2,\ y=-4x=-2 \rightarrow \boxed{(1/2,-2)}$ ↘ $x=-1/2,\ y=-4x=2 \rightarrow \boxed{(-1/2,2)}$

73. $x^2-2xy+y^2-4x=8$

$2x-2x\frac{dy}{dx}-2y+2y\frac{dy}{dx}-4=0$

$\frac{dy}{dx}=\frac{4+2y-2x}{2y-2x}=\frac{2+y-x}{y-x}$

a) Vertical when $y-x=0 \rightarrow y=x$

$x^2-2x(x)+x^2-4x=8$

$\cancel{x^2}-\cancel{2x^2}+\cancel{x^2}-4x=8 \rightarrow x=-2,\ y=x=-2 \rightarrow \boxed{(-2,-2)}$

b) Horizontal when $2+y-x=0 \rightarrow y=x-2$

$x^2-2x(x-2)+(x-2)^2-4x=8$

$\cancel{x^2}-\cancel{2x^2}+\cancel{4x}+\cancel{x^2}-\cancel{4x}+4-4x-8=0 \rightarrow 4x=-4 \rightarrow x=-1,\ y=x-2=-1-2=-3 \rightarrow \boxed{(-1,-3)}$

75. $y=A\sin(Bx+C)+D$

$y'=A\cos(Bx+C)\cdot B=AB\cos(Bx+C)$

If amplitude $=3$, then $A=3$. If period $=\pi$, then $B=2$.

$y'=3\cdot 2\cos(2x+C)=\underbrace{6\cos(2x+C)}_{\text{max value of 1}}=$ at most $6\cdot 1=\boxed{6}$

77. $y=\sin(x-\sin x)$

$y'=\cos(x-\sin x)(1-\cos x)$

Horizontal tangent: $y'=0$, on x-axis: $y=0$

$y'=0$ when $\cos(x-\sin x)=0$ or $1-\cos x=0$

↑ easier to solve

$y'=0$ when $\cos x=1$, so $x=0, 2\pi, 4\pi, \dots, 2k\pi$

$y(2k\pi)=\sin(2k\pi-\sin 2k\pi)=\sin(2k\pi-0)=0$

Yes, slope $=0$ and y value $=0$ at $x=2k\pi$ means horizontal tangents on x-axis.

79. $x^2 + 2xy + 2y^2 = 5$

a) $2x + 2x\frac{dy}{dx} + 2y + 4y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x-2y}{2x+4y} = \frac{-x-y}{x+2y} \text{ at } (1,1) = \frac{-1-1}{1+2} = \boxed{\frac{-2}{3}}$$

b) $$\frac{d^2y}{dx^2} = \frac{(x+2y)(-1-\frac{dy}{dx}) - (-x-y)(1+2\frac{dy}{dx})}{(x+2y)^2} \quad \text{at } \begin{array}{l} x=1 \\ y=1 \\ \frac{dy}{dx} = \frac{-2}{3} \end{array}$$

$$\frac{(1+2)(-1+2/3) - (-1-1)(1+2(-2/3))}{(1+2)^2} = \frac{3(-1/3)-(-2)(-1/3)}{9} = \frac{-1-2/3}{9} = \frac{-5/3}{9} = \boxed{\frac{-5}{27}}$$

81. $f(0) = 2,\ f'(0) = 3,\ f''(0) = -1$

a) $g(x) = e^{kx} + f(x)$

$g'(x) = ke^{kx} + f'(x)$

$g'(0) = ke^{k(0)} + f'(0) = k\cdot e^0 + 3 = k\cdot 1 + 3 = \boxed{k+3}$

$g''(x) = k\cdot ke^{kx} + f''(x)$

$g''(0) = k^2e^0 + f''(0) = k^2\cdot 1 + -1 = \boxed{k^2-1}$

b) $h(x) = \cos(bx)\cdot f(x)$

$h'(x) = \cos(bx)\cdot f'(x) + f(x)\cdot -b\sin(bx)$

$h(0) = \cos 0 \cdot f(0) = 1\cdot 2 = 2 \rightarrow (0,2)$

$h'(0) = \cos(0)\cdot f'(0) + f(0)\cdot -b\sin 0 = 1\cdot 3 + \cancel{2\cdot 0} = 3 \rightarrow \boxed{y = 3x+2}$

83. $f(x) = \ln(1-x^2)$

a) $1-x^2 > 0$

$x^2 < 1$

$-1 < x < 1 \rightarrow \boxed{(-1,1)}$

b) $f'(x) = \frac{1}{1-x^2}\cdot -2x = \boxed{\frac{-2x}{1-x^2}}$

c) $\boxed{(-1,1)}$ If f(x) has a restricted domain, then so does f'(x).

d) $$\frac{(1-x^2)(-2)-(-2x)(-2x)}{(1-x^2)^2} = \frac{-2+2x^2-4x^2}{(1-x^2)^2} = \frac{-2-2x^2}{(1-x^2)^2} = \frac{-2\overbrace{(1+x^2)}^{\text{Positive}}}{(1-x^2)^2} = \frac{-(+)}{(+)} = \boxed{-}$$

Anything squared is positive.