Chapter 4 Review: 2–78 e.o.e.

2. $y = \tan(e^x) \rightarrow y' = \sec^2(e^x) \cdot e^x = \boxed{e^x \sec^2(e^x)}$

6. $s = \cot\left(\frac{2}{t}\right) \rightarrow s' = -\csc^2\left(\frac{2}{t}\right) \cdot \frac{-2}{t^2} = \boxed{\frac{2}{t^2}\csc^2\left(\frac{2}{t}\right)}$

10. $r = \tan^2(3-\theta^2) \rightarrow r' = 2\tan(3-\theta^2)\sec^2(3-\theta^2) \cdot -2\theta = \boxed{-4\theta\tan(3-\theta^2)\sec^2(3-\theta^2)}$

14. $y = xe^{-x} \rightarrow y' = x \cdot e^{-x} \cdot -1 + e^{-x} \cdot 1 = \boxed{-xe^{-x} + e^{-x}}$

18. $r = \log_2(\theta^2) \rightarrow r' = \frac{1}{\theta^2 \cdot \ln 2} \cdot 2\theta = \boxed{\frac{2}{\theta \cdot \ln 2}}$

22. $y = \frac{2x \cdot 2^x}{(x^2+1)^{1/2}} \rightarrow y' = \boxed{\frac{\sqrt{x^2+1}\,(2x \cdot 2^x \ln 2 + 2^x \cdot 2) - 2x \cdot 2^x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{x^2+1}}$

26. $y = (1+t^2)\cot^{-1} 2t \rightarrow y' = (1+t^2) \cdot \frac{-1}{1+4t^2} \cdot 2 + \cot^{-1} 2t \cdot 2t$

$y' = \boxed{\frac{-2-2t^2}{1+4t^2} + 2t\cot^{-1} 2t}$

30. $r = \left(\frac{1+\sin\theta}{1-\cos\theta}\right)^2 \rightarrow r' = 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right) \cdot \frac{(1-\cos\theta)\cos\theta - (1+\sin\theta)\sin\theta}{(1-\cos\theta)^2}$

$r' = \frac{2+2\sin\theta}{1-\cos\theta} \cdot \frac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1-\cos\theta)^2} = \boxed{\frac{2+2\sin\theta}{1-\cos\theta} \cdot \frac{\cos\theta - \sin\theta - 1}{(1-\cos\theta)^2}}$

34. $y = \frac{1}{1-e^x}$

Differentiable when $1 - e^x \neq 0 \rightarrow e^x \neq 1 \rightarrow \boxed{x \neq 0}$

38. $y^2 = \frac{x}{x+1} \rightarrow 2y\frac{dy}{dx} = \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2} \rightarrow 2y\frac{dy}{dx} = \frac{1}{(x+1)^2}$

$\frac{dy}{dx} = \frac{1}{2y(x+1)^2} = \boxed{\frac{1}{2(x+1)^2\sqrt{\frac{x}{x+1}}}}$

42. $x^{1/3}+y^{1/3}=4 \rightarrow \frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}\frac{dy}{dx}=0 \rightarrow \frac{1}{3x^{2/3}}+\frac{1}{3y^{2/3}}\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{-1/(3x^{2/3})}{1/(3y^{2/3})}=\frac{-1}{\cancel{3}x^{2/3}}\cdot\frac{\cancel{3}y^{2/3}}{1}=\frac{-y^{2/3}}{x^{2/3}}=-\left(\frac{y}{x}\right)^{2/3}$$

$$\frac{d^2y}{dx^2}=\boxed{-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\cdot\frac{x\frac{dy}{dx}-y}{x^2}}$$

46. $y=\tan 2x,\ x=\pi/3 \rightarrow y(\pi/3)=\tan 2\pi/3=\frac{+\sqrt{3}/2}{-1/2}=\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{-\cancel{2}}{1}=-\sqrt{3}$

$y'=2\sec^2 2x \rightarrow y'(\pi/3)=2(2)^2=8$

a) Tangent: $\boxed{y+\sqrt{3}=8(x-\pi/3)}$

b) Normal: $\boxed{y+\sqrt{3}=-\frac{1}{8}(x-\pi/3)}$

50. $x=3\cos t,\ y=4\sin t,\ t=3\pi/4$

$x(3\pi/4)=3\cos 3\pi/4=3\cdot-\frac{\sqrt{2}}{2}=\frac{-3\sqrt{2}}{2}$

$y(3\pi/4)=4\sin 3\pi/4=4\cdot\frac{\sqrt{2}}{2}=2\sqrt{2}$

$\left(-\frac{3\sqrt{2}}{2},\ 2\sqrt{2}\right)$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4\cos t}{-3\sin t}=\frac{4\cdot\cancel{-\sqrt{2}/2}}{-3\cdot\cancel{\sqrt{2}/2}}=\frac{4}{3}$$

$y-2\sqrt{2}=\frac{4}{3}\left(x+\frac{3\sqrt{2}}{2}\right)$

$y-2\sqrt{2}=\frac{4}{3}x+2\sqrt{2}$

$\boxed{y=\frac{4}{3}x+4\sqrt{2}}$

54. $f(x)=\begin{cases}\sin 2x, & x\le 0\\ mx, & x>0\end{cases}$

a) $\sin 2x=mx$ at $x=0 \rightarrow \sin 0=m\cdot 0 \rightarrow 0=0 \rightarrow$ $\boxed{\text{Always}}$

b) $2\cos 2x=m$ at $x=0 \rightarrow 2\cos 0=m \rightarrow m=2\cdot 1 \rightarrow \boxed{m=2}$

58. $f(x)=\begin{cases}\sin 2x, & -3\le x<0\\ x^2+2x, & 0\le x\le 3\end{cases}$ Not differentiable at endpoints: $x=\pm 3$

Check $x=0$: $\sin 2x=x^2+2x \rightarrow \sin 0=0+0 \rightarrow 0=0$ ✓ so continuous

$2\cos 2x=2x+2 \rightarrow 2\cos 0=0+2 \rightarrow 2(1)=2$ ✓ so differen.

a) $\boxed{(-3,3)}$

b) $\boxed{x=\pm 3}$

c) $\boxed{\text{None}}$

62. $g'(x) < 0$ for all $x \to$ always decreasing

$y = -2x + 5$ is tangent at $x = 1$

$m = -2$, $(1,3)$

a) Normal: $1/2$ $\to$ $\boxed{y-3 = \frac{1}{2}(x-1)}$

b) $g(g^{-1}(x)) = x \to g'(g^{-1}(x)) \cdot (g^{-1}(x))' = 1 \to (g^{-1}(x))' = \dfrac{1}{g'(g^{-1}(x))}$

$1 \underset{g^{-1}}{\overset{g}{\rightleftarrows}} 3$

$g^{-1}(x) = 1$ at $x = 3$

$(g^{-1}(3))' = \dfrac{1}{g'(g^{-1}(3))} = \dfrac{1}{g'(1)} = \dfrac{1}{-2} = \dfrac{-1}{2} \to \boxed{y-1 = \frac{-1}{2}(x-3)}$

c) $y = g(x^2)$ at $x = 1 \to y(1) = g(1) = 3 \to (1,3)$ $\to \boxed{y-3 = -4(x-1)}$

$y' = g'(x^2) \cdot 2x \to y'(1) = 2g'(1) = 2 \cdot -2 = -4$

66. a) $\sqrt{x}\, f(x)$ at $x = 1$

$\sqrt{x}\, f'(x) + f(x) \cdot \dfrac{1}{2\sqrt{x}} \to \sqrt{1} \cdot \dfrac{1}{5} + -3 \cdot \dfrac{1}{2\sqrt{1}} = \dfrac{1}{5} - \dfrac{3}{2} = \dfrac{2}{10} - \dfrac{15}{10} = \boxed{\dfrac{-13}{10}}$

b) $\sqrt{f(x)}$ at $x = 0$

$\frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}} \to \dfrac{-\cancel{2}}{\cancel{2}\sqrt{9}} = \boxed{\dfrac{-1}{3}}$

c) $f(\sqrt{x})$ at $x = 1$

$f'(\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}} \to f'(1) \cdot \dfrac{1}{2\cancel{\sqrt{1}}} = \dfrac{1}{2} \cdot \dfrac{1}{5} = \boxed{\dfrac{1}{10}}$

d) $f(1 - 5\tan x)$ at $x = 0$

$f'(1 - 5\tan x) \cdot -5\sec^2 x \to f'(1) \cdot -5 \cdot 1^2 = \frac{1}{5} \cdot -5 = \boxed{-1}$

e) $\dfrac{f(x)}{2 + \cos x}$ at $x = 0$

$\dfrac{(2+\cos x) f'(x) + f(x)\sin x}{(2+\cos x)^2} \to \dfrac{(2+1)(-2) + \cancel{9(0)}}{(2+1)^2} = \dfrac{-6}{9} = \boxed{\dfrac{-2}{3}}$

f) $10\sin(\frac{\pi}{2}x) \cdot (f(x))^2$ at $x = 1$

$10\sin(\frac{\pi}{2}x) \cdot 2f(x)f'(x) + (f(x))^2 + 5\pi\cos(\frac{\pi}{2}x)$

$10(1) \cdot 2(-3) \cdot \frac{1}{5} + \cancel{(-3)^2 \cdot 5\pi \cdot 0} = \boxed{-12}$

70. $s(t) = 10\cos(t + \pi/4)$

a) $x(t) = 10\cos(t + \pi/4)$

$y(t) = 0$

b) $s(0) = 10\cos(\pi/4) = 10 \cdot \frac{\sqrt{2}}{2} = \boxed{5\sqrt{2}}$

c) Max & min of $10\cos(t + \pi/4)$ are $\boxed{\pm 10}$

d) $10\cos(t+\pi/4) = 0 \to \cos(t+\pi/4) = 0 \to t + \pi/4 = \pi/2 \to \boxed{t = \pi/4}$

$v(t) = -10\sin(t + \pi/4) \to v(\pi/4) = -10\sin\pi/2 = -10(1) = \boxed{-10}$

speed $= |\text{velocity}| = |-10| = \boxed{10}$

$a(t) = -10\cos(t + \pi/4) \to a(\pi/4) = -10\cos\pi/2 = -10(0) = \boxed{0}$

74. $x^2 - 2xy + y^2 - 4x = 8$

$x = 0$: $y^2 = 8 \to y = \pm\sqrt{8} \to (0, \sqrt{8}), (0, -\sqrt{8})$

$y = 0$: $x^2 - 4x = 8 \to x^2 - 4x - 8 = 0 \to a = 1, b = -4, c = -8$

$$x = \frac{4 \pm \sqrt{4^2 - 4(1)(-8)}}{2(1)} = \frac{4 \pm \sqrt{48}}{2} = \frac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}$$

$(2+2\sqrt{3}, 0), (2-2\sqrt{3}, 0)$

$$2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} - 4 = 0 \to \frac{dy}{dx} = \frac{2y - 2x + 4}{2y - 2x} = \frac{y - x + 2}{y - x}$$

At $(0, \sqrt{8})$: $\dfrac{\sqrt{8} - \cancel{0} + 2}{\sqrt{8} - \cancel{0}} = \dfrac{\sqrt{8} + 2}{\sqrt{8}} = \dfrac{2\sqrt{2} + 2}{2\sqrt{2}} = \boxed{\dfrac{\sqrt{2} + 1}{\sqrt{2}}}$

At $(0, -\sqrt{8})$: $\dfrac{-\sqrt{8} - \cancel{0} + 2}{-\sqrt{8} - \cancel{0}} = \dfrac{-\sqrt{8} + 2}{-\sqrt{8}} = \dfrac{-2\sqrt{2} + 2}{-2\sqrt{2}} = \boxed{\dfrac{-\sqrt{2} + 1}{-\sqrt{2}}}$

At $(2+2\sqrt{3}, 0)$: $\dfrac{\cancel{0} - \cancel{2} - 2\sqrt{3} + \cancel{2}}{\cancel{0} - 2 - 2\sqrt{3}} = \dfrac{-2\sqrt{3}}{-2-2\sqrt{3}} = \boxed{\dfrac{\sqrt{3}}{1 + \sqrt{3}}}$

At $(2-2\sqrt{3}, 0)$: $\dfrac{\cancel{0} - \cancel{2} + 2\sqrt{3} + \cancel{2}}{\cancel{0} - 2 + 2\sqrt{3}} = \dfrac{2\sqrt{3}}{-2+2\sqrt{3}} = \boxed{\dfrac{\sqrt{3}}{-1 + \sqrt{3}}}$

78. $P(t) = \dfrac{200}{1+e^{5-t}}$

a) $P(0) = \dfrac{200}{1+e^{5}} = 1.339 \rightarrow \boxed{\text{1student}}$

b) $\lim\limits_{t\to\infty} P(t) = \dfrac{200}{1+e^{5-\infty}} = \dfrac{200}{1+e^{-\infty}} = \dfrac{200}{1+\frac{1}{e^{\infty}}} = \dfrac{200}{1+0} = \boxed{200 \text{ stud.}}$

c) Greatest growth rate at $\frac{1}{2}$ of carrying capacity

$\frac{1}{2}(200) = 100$

$\dfrac{200}{1+e^{5-t}} = 100 \rightarrow \dfrac{200}{100} = 1+e^{5-t} \rightarrow 1+e^{5-t} = 2 \rightarrow e^{5-t} = 1$

$(5-t)\cancel{\ln e} = \ln 1 \rightarrow 5-t = 0 \rightarrow \boxed{t = 5 \text{ days}}$

$P(t) = 200(1+e^{5-t})^{-1}$

$P'(t) = \cancel{-}200(1+e^{5-t})^{-2} \cdot e^{5-t} \cdot \cancel{-1} = \dfrac{200e^{5-t}}{(1+e^{5-t})^{2}}$

$P'(5) = \dfrac{200\cancel{e}^{0}}{(1+e^{0})^{2}} = \dfrac{200}{2^{2}} = \dfrac{200}{4} = \boxed{50 \text{ students/day}}$