# Curve Sketching : Day 1

1. $f(x) = x^{4/3} + 4x^{1/3}$

a) Tangent horizontal: $f'(x) = 0$

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = 0$$

$$x^{1/3} + x^{-2/3} = 0$$

$$x^{2/3} \cdot x^{1/3} = \frac{-1}{\cancel{x^{2/3}}} \cdot \cancel{x^{2/3}}$$

$$\boxed{x = -1}$$

$$f(-1) = (-1)^{4/3} + 4(-1)^{1/3} = 1 + 4(-1) = 1 - 4 = -3$$

Coordinates: $\boxed{(-1,-3)}$

b) Tangent vertical: $f'(x)$ DNE (0 in denominator)

$$f'(x) = \frac{4x^{1/3}}{3} + \frac{4}{3x^{2/3}} \; : \; f'(x) \text{ DNE when } 3x^{2/3} = 0 \rightarrow x = 0$$

$f(0) = 0^{4/3} + 4 \cdot 0^{1/3} = 0 + 0 = 0$, so coordinates: $\boxed{(0,0)}$

c) Domain: $(-\infty, \infty)$, so no endpoints to check

Max/min when $f'(x) = 0$ or $f'(x)$ DNE $\rightarrow$ $x = -1$, $x = 0$ (from a & b)

$f'(x)$: $- - - \; 0 \; + + + \; 0 \; + + +$

$x = -1$: Min; $x = 0$: Neither

$$f'(x) = \frac{4}{3}\left(\sqrt[3]{x} + \frac{1}{x^{2/3}}\right)$$

$$f'(1) = \frac{4}{3}\left(\sqrt[3]{1} + \frac{1}{1^{2/3}}\right) = \frac{4}{3}(1+1) = (+)(+) = +$$

$$f'(-8) = \frac{4}{3}\left(\sqrt[3]{-8} + \frac{1}{(-8)^{2/3}}\right) = \frac{4}{3}\left(-2 + \frac{1}{4}\right) = (+)(-) = -$$

$$f'(-1/8) = \frac{4}{3}\left(\sqrt[3]{-\frac{1}{8}} + \frac{1}{(-1/8)^{2/3}}\right) = \frac{4}{3}\left(-\frac{1}{2} + \frac{1}{1/4}\right) = \frac{4}{3}\left(-\frac{1}{2} + 4\right)$$

$$= (+)(+) = +$$

$$\boxed{\begin{array}{l} \text{Abs. Min} : (-1,-3) \\ \text{Abs. Max} : \text{DNE} \end{array}}$$

d) $f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$

$$f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9x^{2/3}} - \frac{8}{9x^{5/3}} \quad \text{DNE when } x = 0$$

$$\frac{4}{9x^{2/3}} - \frac{8}{9x^{5/3}} = 0$$

$$\frac{4}{\cancel{9}x^{2/3}} = \frac{8}{\cancel{9}x^{5/3}}$$

$$\frac{4x^{5/3}}{x^{2/3}} = \frac{8\cancel{x^{2/3}}}{\cancel{x^{2/3}}}$$

$$4x = 8$$

$$x = 2$$

$f''(x)$: $+ + + \; 0 \; - - - - - \; 0 \; + + + +$

$x = 0$, $x = 2$

$$f''(x) = \frac{4}{9x^{2/3}} - \frac{8}{9x^{5/3}}$$

CC down: $\boxed{(0,2)}$

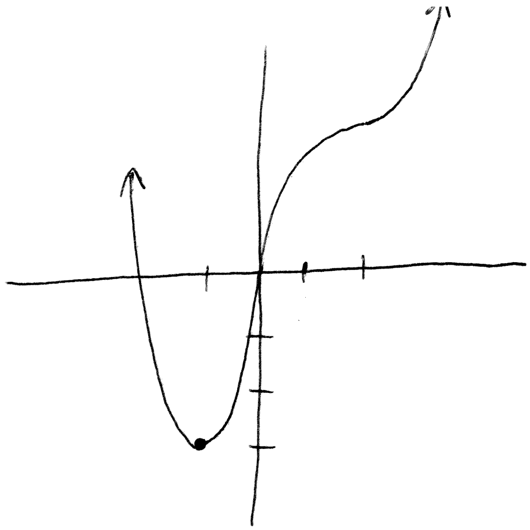
$$f''(-1) = \frac{4}{9(-1)^{2/3}} - \frac{8}{9(-1)^{5/3}} = \frac{4}{9} + \frac{8}{9} = +$$

$$f''(1) = \frac{4}{9} - \frac{8}{9} = -$$

$$f''(3) = \frac{4}{9\sqrt[3]{9}} - \frac{8}{9\sqrt[3]{243}} = \frac{4}{9}\left(\frac{1}{\sqrt[3]{9}} - \frac{2}{\sqrt[3]{243}}\right) = \frac{4}{9}\left(\frac{1}{\sqrt[3]{9}} - \frac{2}{3\sqrt[3]{9}}\right)$$

$$f''(3) = \frac{4}{9\sqrt[3]{9}}\left(1 - 2/3\right) = (+)(+) = +$$

1. e)



2. $f(x) = |x| \cdot 0.5e^{-x^2} = \dfrac{0.5|x|}{e^{x^2}}$

$x = \pm 1$ for example:

a) $x = 1$: $\dfrac{0.5 \cdot |1|}{e^{1^2}} = \dfrac{0.5}{e^1} = \dfrac{0.5}{e} \rightarrow (1, 0.5/e)$ $\quad f(-x) = f(x) \rightarrow \boxed{\text{Even}}$

(symmetric to y-axis)

$x = -1$: $\dfrac{0.5|-1|}{e^{(-1)^2}} = \dfrac{0.5}{e^1} = \dfrac{0.5}{e} \rightarrow (-1, 0.5/e)$

b) For $x > 0$ (start with positive side, then make left side symmetric)

$f(x) = 0.5x \cdot e^{-x^2}$ (bc $|x| = x$ when $x > 0$)

$f'(x) = \cancel{0.5}x \cdot e^{-x^2} \cdot -\cancel{2}x + e^{-x^2} \cdot 0.5 = -x^2e^{-x^2} + 0.5e^{-x^2} = e^{-x^2}(-x^2 + 0.5)$

$\dfrac{1}{e^{x^2}}(0.5 - x^2) = 0 \rightarrow \dfrac{1}{e^{x^2}} \neq 0$ never bc $e^u$ is always +

$0.5 - x^2 = 0 \rightarrow x^2 = 1/2 \rightarrow x = \sqrt{1/2} = 1/\sqrt{2} = \sqrt{2}/2 \approx 1.4/2 = 0.7$

(estimate for sign line

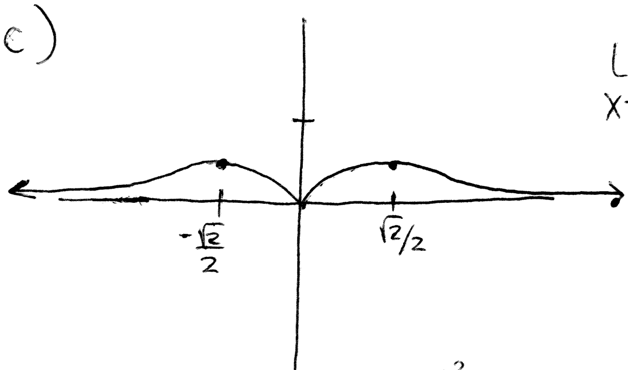
$\underset{x=0}{|}\; + + + \overset{0}{\underset{x=\frac{\sqrt{2}}{2}}{|}} - - -$ $\quad f'(x) = \dfrac{1}{e^{x^2}}(1/2 - x^2)$ for $x > 0$

$f'(1/2) = \dfrac{1}{e^{1/4}}(1/2 - 1/4) = (+)(+) = +$

$f'(1) = \dfrac{1}{e}(1/2 - 1) = (+)(-) = -$

Inc: $\boxed{(0, \sqrt{2}/2) \cup (-\infty, -\sqrt{2}/2)}$

c)




$\lim\limits_{x \to \infty} \dfrac{0.5|x|}{e^{x^2}} = \dfrac{0.5\infty}{e^{\infty^2}}$ ← big ← much bigger $= 0$

Min: $(0, 0)$

Max: $\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\sqrt{\frac{2}{e}}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{1}{4}\sqrt{\frac{2}{e}}\right)$

$f(\sqrt{2}/2) = \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2} \cdot e^{-(\sqrt{2}/2)^2} = \dfrac{\sqrt{2}}{4} \cdot e^{-1/2} = \dfrac{\sqrt{2}}{4\sqrt{e}} = \dfrac{1}{4}\sqrt{\dfrac{2}{e}}$

3. $y = x + \sin x$ on $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ (endpoints included)

a) $y(-\pi/2) = -\frac{\pi}{2} + \sin(-\pi/2) = -\frac{\pi}{2} - 1 \rightarrow$ $\left(-\frac{\pi}{2}, -\frac{\pi}{2} - 1\right)$: Abs. Min

$y(3\pi/2) = \frac{3\pi}{2} + \sin(3\pi/2) = \frac{3\pi}{2} - 1 \rightarrow$ $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$: Abs. Max

$y' = 1 + \cos x$ exists for all $x$ (never DNE)

$1 + \cos x = 0 \rightarrow \cos x = -1$ at $x = \pi$

$y(\pi) = \pi + \sin\pi = \pi + 0 = \pi \rightarrow (\pi, \pi)$

Neither

$f'(x)$: $-\frac{\pi}{2}$ + + + + + 0 ($x = \pi$) + + $\frac{3\pi}{2}$ $1+\cos x$

$f'(0) = 1 + \cos 0 = 1 + 1 = 2$ (+)

$f'(5\pi/4) = 1 - \frac{\sqrt{2}}{2} \approx 1 - \frac{1.4}{2} = 1 - 0.7 \approx 0.3$ (+

b) $y' = 1 + \cos x$

$y'' = -\sin x$ exists for all $x$ (never DNE)

$-\sin x = 0 \rightarrow \sin x = 0 \rightarrow x = 0, x = \pi$

$y(0) = 0 + \sin 0 = 0 + 0 = 0 \rightarrow \boxed{(0,0)}$

$y(\pi) = \pi + \sin\pi = \pi + 0 = \pi \rightarrow \boxed{(\pi, \pi)}$

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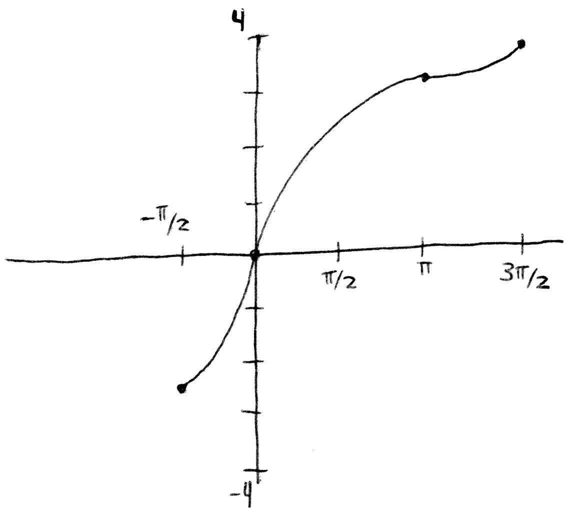
$f''(x)$: $-\frac{\pi}{2}$ + + + 0 ($x=0$) − − − − 0 ($x=\pi$) + + + $\frac{3\pi}{2}$ $-\sin x$

c)




$-\frac{\pi}{2} - 1 \approx \frac{-3.1}{2} - 1 = -1.55 - 1 = -2.55$

$\frac{3\pi}{2} - 1 \approx \frac{3(3.1)}{2} - 1 = \frac{9.3}{2} - 1 = 4.65 - 1 = 3.65$

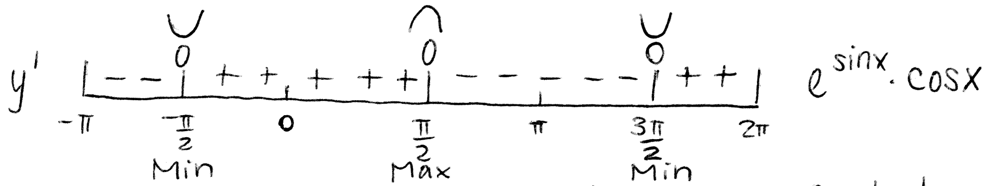
4. $y = e^{\sin x}$ on $-\pi < x < 2\pi$ (no endpoints)

a) $y' = e^{\sin x} \cdot \cos x$ exists for all $x$ (never DNE)

$e^{\sin x} \cdot \cos x = 0$ when $e^{\sin x} = 0$ or $\cos x = 0$

$e^{\sin x} = 0 \rightarrow$ never

$\cos x = 0 \rightarrow x = -\pi/2,\ \pi/2,\ 3\pi/2$




$e^{\sin x}$ is always positive, so then sign of $y'$ depends on $\cos x$ being + or -

$y(-\pi/2) = e^{\sin(-\pi/2)} = e^{-1} = 1/e \rightarrow \boxed{(-\frac{\pi}{2}, \frac{1}{e}) \text{ Min}}$

$y(\pi/2) = e^{\sin \pi/2} = e^{1} = e \rightarrow \boxed{(\pi/2, e) \text{ Max}}$

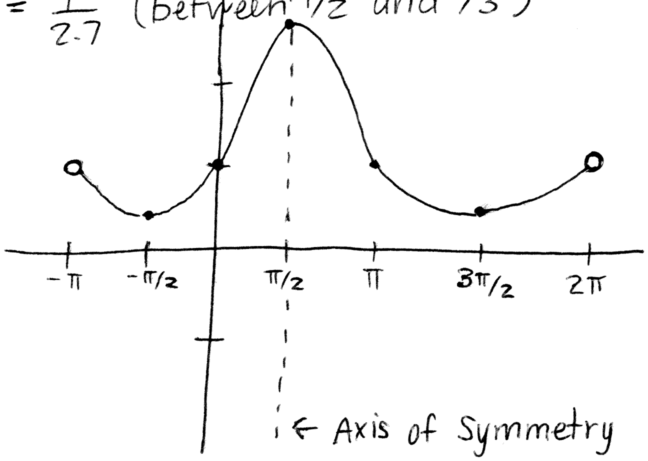
$y(3\pi/2) = e^{\sin 3\pi/2} = e^{-1} = 1/e \rightarrow \boxed{(3\pi/2, \frac{1}{e}) \text{ Min}}$

b) $e \approx 2.7$

$\frac{1}{e} = \frac{1}{2.7}$ (between 1/2 and 1/3)

$y(-\pi) = e^{\sin(-\pi)} = e^{0} = 1 \rightarrow (-\pi, 1)$

$y(2\pi) = e^{\sin 2\pi} = e^{0} = 1 \rightarrow (2\pi, 1)$




c) $\boxed{x = \pi/2}$

5. $f(x) = (x^2-1)^3$ for all $x$ $(\mathbb{R})$

a) $f'(x) = 3(x^2-1)^2 \cdot 2x = 6x(x^2-1)^2 = 0$ when $6x = 0$ or $(x^2-1)^2 = 0$

$x = 0$ $\quad x^2 - 1 = 0$

$x = \pm 1$

Neither $\quad$ ∪ $\quad$ Neither

$f'(x)$ $\quad - - 0 - - 0 + + 0 + + +$ $\quad 6x(x^2-1)^2$

$x=-1 \quad x=0 \quad x=1$

Min

$(x^2-1)^2$ is always positive, so the sign of $f'(x)$ only depends on $6x$.

Inc: $\boxed{(0,1) \cup (1,\infty)}$

b) $f(0) = (0^2-1)^3 = (-1)^3 = -1 \rightarrow \boxed{(0,-1) \text{ Min}}$ (sign line in part a)

c) $f'(x) = 6x(x^2-1)^2$

$f''(x) = 6x \cdot 2(x^2-1) \cdot 2x + (x^2-1)^2 \cdot 6 = 24x^2(x^2-1) + 6(x^2-1)^2$

$f''(x) = (x^2-1)(24x^2 + 6(x^2-1)) = (x^2-1)(24x^2+6x^2-6) = (x^2-1)(30x^2-6)$

$(x^2-1)(30x^2-6) = 0$

$x^2 - 1 = 0 \quad 30x^2 - 6 = 0$

$x^2 = 1 \quad x^2 = 6/30 = 1/5$

$x = \pm 1 \quad x = \pm\sqrt{1/5}$

$f''(x)$ $\quad + + 0 - - - 0 + + + 0 - - - 0 + + +$

$-1 \quad -\sqrt{\frac{1}{5}} \quad \sqrt{\frac{1}{5}} \quad 1$

$f''(x) = (x^2-1)(30x^2-6)$

$f''(0) = (-1)(-6) = +$

$f''(2) = (4-1)(120-6) = (+)(+) = + = f''(-2)$

$f''\left(\sqrt{\frac{1}{4}}\right) = \left(\frac{1}{4}-1\right)(30/4 - 6) = (-)(+) = - = f''\left(-\sqrt{\frac{1}{4}}\right)$

CC up:

$\boxed{(-\infty,-1) \cup \left(-\sqrt{\frac{1}{5}}, \sqrt{\frac{1}{5}}\right) \cup (1,\infty)}$

d) $\sqrt{\frac{1}{9}} < \sqrt{\frac{1}{5}} < \sqrt{\frac{1}{4}}$

$\frac{1}{3} < \sqrt{\frac{1}{5}} < \frac{1}{2}$

$(x^2-1)^3 = 0$

$x^2 - 1 = 0$

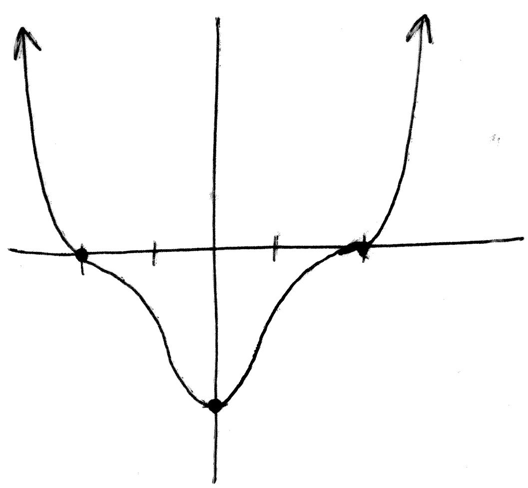
$x^2 = 1$

$x = \pm 1$

Roots

$(1,0)$

$(-1,0)$



6. $y = \cos x - \cos^2 x$ on $-\pi < x < \pi$ (no endpoints)

a) x-int when $y=0$ : $\cos x - \cos^2 x = 0$

$$\cos x(1-\cos x) = 0$$

$$\cos x = 0 \quad \text{or} \quad 1-\cos x = 0 \rightarrow \cos x = 1$$

$$\boxed{x = -\tfrac{\pi}{2}, \tfrac{\pi}{2}} \qquad \boxed{x=0}$$

b) $y' = -\sin x - 2\cos x(-\sin x) = -\sin x + 2\cos x \sin x = \sin x(2\cos x - 1)$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $2\cos x - 1 = 0$

$x = 0$ $\qquad \cos x = 1/2$ at $x = \pi/3, -\pi/3$

$y'$: $-\pi$ | $+ + +$ | 0 ($x=-\frac{\pi}{3}$, Max) | $- -$ | 0 ($x=0$, Min) | $+ +$ | 0 ($x=\frac{\pi}{3}$, Max) | $- - -$ | $\pi$ $\qquad \sin x(2\cos x - 1)$

$y'(-\pi/2) = -1(2\cdot 0 - 1) = -1(0-1) = -1\cdot -1 = 1 = +$

$y'(-\pi/4) = -\frac{\sqrt{2}}{2}\left(\cancel{2}\frac{\sqrt{2}}{\cancel{2}} - 1\right) = (-)(+) = -$

$y'(\pi/4) = \frac{\sqrt{2}}{2}\left(\cancel{2}\frac{\sqrt{2}}{\cancel{2}} - 1\right) = (+)(+) = +$

$y'(\pi/2) = 1(2\cdot 0 - 1) = 1(-1) = -$

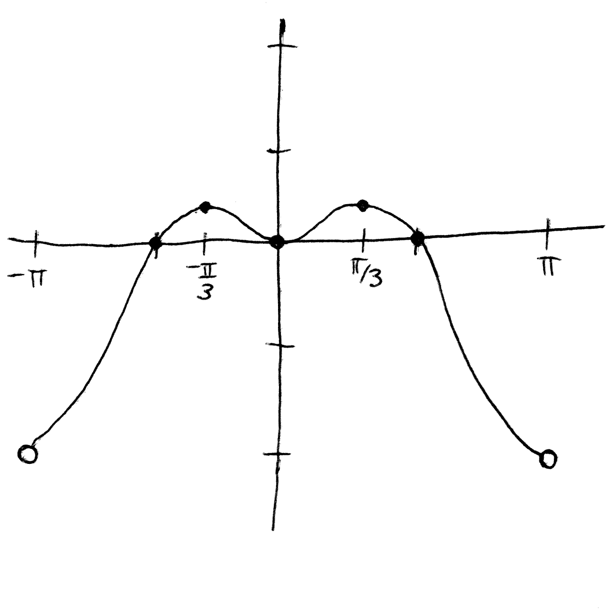
$y(-\pi/3) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \rightarrow \boxed{\left(-\frac{\pi}{3}, \frac{1}{4}\right) \text{Max}}$

$y(0) = 1 - 1^2 = 1 - 1 = 0 \rightarrow (0,0)$ Min

$y(\pi/3) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \rightarrow \boxed{\left(\frac{\pi}{3}, \frac{1}{4}\right) \text{Max}}$

c) Inc: $\boxed{(-\pi, -\pi/3) \cup (0, \pi/3)}$

d)




$y(-\pi) = -1 - (-1)^2 = -1 - 1 = -2 = y(\pi)$