Differential Equations Packet

#1 a) $\frac{dy}{dx} = 6x^2 - x^2y$; $f(-1) = 2$

Point	Slope	Δx	$\Delta y = \frac{dy}{dx} \cdot \Delta x$	New Point
$(-1, 2)$	$6(-1)^2 - (-1)^2 2$ $= 6 - 2$ $= 4$	$.5$	$= 4(.5)$ $= 2$	$(-.5, 4)$
$(-.5, 4)$	$6(-.5)^2 - (-.5)^2(4)$ $= .5$	$.5$	$= .5(.5)$ $= .25$	$(0, 4.25)$

$\boxed{\therefore f(0) \approx 4.25}$

b) $\frac{d^2y}{dx^2} = -12$

$f(-1) = 2$
$f'(-1) = 4$
$f''(-1) = -12$

$\therefore T_2(x) = 2 + 4(x+1) - \frac{12}{2}(x+1)^2$

$= 2 + 4x + 4 - 6(x^2 + 2x + 1)$

$\boxed{T_2(x) = -6x^2 - 8x}$

c) $\frac{dy}{dx} = x^2(6-y)$

$\int \frac{dy}{6-y} = \int x^2\,dx$

$-\ln|6-y| = \frac{x^3}{3} + C$

$\ln|6-y| = -\frac{x^3}{3} + C$

$\ln|6-2| = -\frac{1}{3}(-1) + C$

$\ln 4 = \frac{1}{3} + C$

$C = \ln 4 - \frac{1}{3}$

$e^{\ln|6-y|} = e^{-\frac{x^3}{3} + \ln 4 - \frac{1}{3}}$

$6 - y = 4e^{-\frac{x^3}{3} - \frac{1}{3}}$

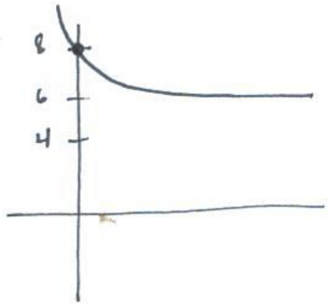
$\boxed{6 - 4e^{-\frac{x^3}{3} - \frac{1}{3}} = y}$ OR $\boxed{y = 6 - 4^{-1/3(x^3+1)}}$

$6 - 4e^{-\frac{x^3}{3}} - 4e^{-1/3} = y$

$6 - 4e^{-\frac{x^3}{3}} - 2.866\ldots = y$

$\boxed{3.134 - 4e^{-\frac{x^3}{3}} = y}$

#2. $\frac{dy}{dt} = \frac{y}{8}(6-y)$; $f(0)=8$

a) 

b)

Point	Slope	Δx	$\Delta y = \frac{dy}{dx}\cdot\Delta x$	New Point
$(0,8)$	$\frac{8}{8}(6-8) = -2$	$.5$	$\Delta y = -2(.5) = -1$	$(.5, 7)$
$(.5, 7)$	$\frac{7}{8}(6-7) = -\frac{7}{8}$	$.5$	$\Delta y = -\frac{7}{8}(.5) = -.4375$	$(1, 6.5625)$ $\left(1, \frac{105}{16}\right)$

New Point $\rightarrow$ $f(1) = 6.5625$

c) $f(0) = 8 \rightarrow b_0$

$f'(0) = -2 \rightarrow b_1$

$f''(0) = \frac{3}{4} - \frac{1}{4}(8) = \frac{-5}{4} \rightarrow 2b_2$

$b_2 = -\frac{5}{8}$

$f'(t) = \frac{y}{8}(6-y)$

$f'(t) = \frac{3}{4}y - \frac{y^2}{8}$

$f''(t) = \frac{3}{4} - \frac{1}{4}y$

$Q(x) = 8 - 2(x-0) - \frac{5}{8}(x-0)^2$

$\boxed{Q(x) = 8 - 2x - \frac{5}{8}x^2}$

$Q(1) = 8 - 2 - \frac{5}{8} = 5.375$

d) Range: $(6, \infty)$ (from the slope field)

3a) $\frac{dy}{dx} = 3x + 2y + 1$

$$\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2(3x+2y+1)$$
$$= 3 + 6x + 4y + 2$$
$$\boxed{= 6x + 4y + 5}$$

#3 b) $y = mx + b + e^{rx}$

$y' = m + re^{rx} = 3x + 2y + 1$

$m + re^{rx} = 3x + 2(mx + b + e^{rx}) + 1$

$0x + m + re^{rx} = 3x + 2mx + 2b + 2e^{rx} + 1$

$0 = 3 + 2m$

$-3 = 2m$

$\boxed{m = -\frac{3}{2}}$

$m = 2b + 1$

$-\frac{3}{2} = 2b + 1$

$-\frac{5}{2} = 2b$

$\boxed{-\frac{5}{4} = b}$

$re^{rx} = 2e^{rx}$

$\boxed{r = 2}$

c)

Point	Slope ($3x+2y+1$)	Δx	$\Delta y = \frac{dy}{dx} \cdot \Delta x$	New Point
$(0,-2)$	$= 3(0)+2(-2)+1$ $= -3$	$.5$	$= -3(.5)$ $= -1.5$	$(.5, -3.5)$
$(.5,-3.5)$	$= 3(.5)+2(-3.5)+1$ $= -4.5$	$.5$	$= -4.5(.5)$ $= -2.25$	$\boxed{(1, -5.75)}$

d) $g(0) = K$

$(0, K)$ slope $= 3(0) + 2K + 1$ $= 2K + 1$ $\Delta x = 1$ $\Delta y = 2K + 1$ New Point: $(1, K + 2k + 1)$

$(1, 3k + 1)$

$3k + 1 = 0$

$\boxed{k = -1/3}$

#4 a) $f(4)=1$ $\quad \frac{dy}{dx} = 2y(3-x)$

$$\int \frac{dy}{2y} = \int (3-x)\,dx$$

$$\frac{1}{2}\ln|2y| = 3x - \frac{1}{2}x^2 + c$$

$$\ln|2y| = 6x - x^2 + c$$

plug in $(4,1) \rightarrow \ln|2| = 6(4) - (4)^2 + c$

$$\ln(2) - 8 = c$$

$$e^{\ln|2y|} = e^{6x-x^2+\ln 2-8}$$

$$2y = 2e^{6x-x^2-8}$$

$$\boxed{y = e^{-x^2+6x-8}}$$

OR $y = 3.355\times10^{-4} e^{6x-x^2}$

b) $\frac{dy}{dx} = 2y(3-y) \rightarrow$ LOGISTIC!

$= 6y - 2y^2$

$\therefore$ carrying capacity $= 3$

so $\lim_{x\to\infty} g(x) = 3$ and $\lim_{x\to\infty} g'(x) = 0$

the slope is approaching zero as it gets close to the carrying capacity.

c) $y' = 6y - 2y^2$

$y'' = (6-4y)\frac{dy}{dx}$

$\boxed{y = 3/2}$ $\quad 2y(3-y)$

$\boxed{y=0 \quad y=3} \rightarrow$ these are horz. asym.

$\therefore$ not points of inflection

* Also... points of inflection are half of the carrying capacity

$\boxed{\begin{array}{l} \therefore \text{ ccap.} = 3 \\ \text{pt of infl.} = 3/2 \end{array}$

#5 a) $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$; $f(-1) = -4$

$= 5(-1)^2 - \frac{6}{-4-2}$

$= 5+1$

$\boxed{\frac{dy}{dx} = 6}$

$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$

$= 10(-1) + 6(-4-2)^{-2}(6)$

$= -10 + \frac{6}{36}(6)$

$\boxed{\frac{d^2y}{dx^2} = -9}$

b) x-axis → $y=0$ and $=0$
OR
$(K, 0)$

$0 = 5K^2 - \frac{6}{0-2}$

$0 = \underbrace{5K^2 + 3}_{\text{always }(+)}$ ∴ No!

c) $f(-1) = -4 \rightarrow b_0$

$f'(-1) = 6 \rightarrow b_1$

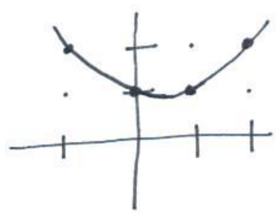
$f''(-1) = -9 \rightarrow 2b_2$

$b_2 = \frac{-9}{2}$

$\boxed{Q(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2}$

#6 $\frac{dy}{dx} = 2x - y$

a)



b) Thru $(0,1)$

min @ $x = \ln\left(\frac{3}{2}\right)$

↳ $y' = 0$

$0 = 2\left[\ln(3/2)\right] - y$

$\boxed{y = 2\ln(3/2)}$

c) $f(0) = 1$ $\frac{dy}{dx} = 2x - y$

Point	Slope	Δx	$\Delta y = \Delta x \cdot \frac{dy}{dx}$	New Point
$(0,1)$	$2(0)-1 = -1$	$-.2$	$\Delta y = -1(-.2) = +.2$	$(-.2, 1.2)$
$(-.2, 1.2)$	$2(-.2)-(1.2) = -1.6$	$-.2$	$\Delta y = -1.6(-.2) = .32$	$\boxed{(-.4, 1.52)}$

#6d) $\frac{dy}{dx} = 2x - y$

$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$= 2 - (2x - y)$

$\frac{d^2y}{dx^2} = 2 - 2x + y$ @ $(-4, 1.52)$

$= 2 - 2(-4) + 1.52 \rightarrow$ positive $\therefore$ cc up so the approx. is above the actual graph.

#7 $\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right)$

carrying capacity

a) $\therefore \lim_{t \to \infty} P(t) = 12$ for both

b) Half of C.C. $\rightarrow$ 6

c) $\frac{dy}{dt} = \frac{y}{5}\left(1 - \frac{t}{12}\right)$; $y(0) = 3$

$\frac{5}{y}\,dy = \left(1 - \frac{t}{12}\right)dt$

$5\ln|y| = t - \frac{t^2}{24} + C$

$5\ln|3| = C$

$\frac{5\ln|y|}{5} = \frac{t - \frac{t^2}{24} + 5\ln(3)}{5}$

$e^{\ln|y|} = e^{\frac{t}{5} - \frac{t^2}{120}} + e^{\ln 3}$

$\boxed{y = 3e^{t/5 - t^2/120}}$

d) $\lim_{t \to \infty} 3e^{t/5 - t^2/120} = 0$

#8 $V = \pi r^2 h$ $\quad \frac{dV}{dt} = -5\pi\sqrt{h}$

a) $V = \pi(5)^2 h$

$V = 25\pi h$

$\frac{dV}{dt} = 25\pi \frac{dh}{t}$

$\frac{-5\pi\sqrt{h}}{25\pi} = \frac{25\pi \frac{dh}{dt}}{25\pi}$

$\boxed{\frac{-\sqrt{h}}{5} = \frac{dh}{dt}}$

b) $\frac{dh}{dt} = \frac{-\sqrt{h}}{5}$ $\qquad h=17$ @ $t=0$

$\int \frac{dh}{\sqrt{h}} = \int \frac{-1}{5} dt$

$2h^{1/2} = \frac{-1}{5}t + C$

$2\sqrt{17} = \frac{-1}{5}(0) + C$

$C = 2\sqrt{17}$

$2\sqrt{h} = \frac{-1}{5}t + 2\sqrt{17}$

$\sqrt{h} = \frac{-1}{10}t + \sqrt{17}$

$\boxed{h = \left(\frac{-1}{10}t + \sqrt{17}\right)^2}$

c) coffeepot empty $\rightarrow h = 0$

$\sqrt{0} = \sqrt{\left(\frac{-1}{10}t + \sqrt{17}\right)^2}$

$0 = \frac{-1}{10}t + \sqrt{17}$

$\frac{1}{10}t = \sqrt{17}$

$\boxed{t = 10\sqrt{17}}$

#9 $\frac{dy}{dx} = \frac{3-x}{y}$

a) $y=-2$ is tangent $\rightarrow \frac{dy}{dx} = 0$

$0 = \frac{3-x}{y}$

$0 = 3-x$

$x = 3 \rightarrow \boxed{(3,-2)}$

$\frac{d^2y}{dx^2} = \frac{y(-1) - (3-x)\frac{dy}{dx}}{y^2}$

$= \frac{(-2)(-1) - \cancel{(3-3)\frac{dy}{dx}}}{(-2)^2}$

$= \frac{2}{4} = \frac{1}{2} \rightarrow$ cc up

$\therefore$ the point is a MIN!

b) $g(6) = -4$

$\frac{dy}{dx} = \frac{3-x}{y}$

$y\,dy = (3-x)dx$

$\frac{y^2}{2} = 3x - \frac{x^2}{2} + C$

$\frac{(-4)^2}{2} = 3(6) - \frac{(6)^2}{2} + C \rightarrow 8 = 18 - 18 + C$

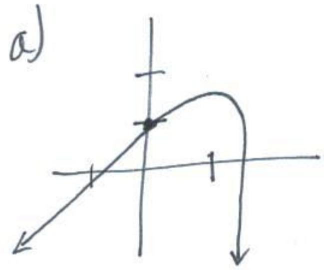
$C = 8$

$\frac{y^2}{2} = 3x - \frac{x^2}{2} + 8$

$y^2 = 6x - x^2 + 16$

$\boxed{y = \pm\sqrt{6x - x^2 + 16}}$

#10 $\frac{dy}{dx} = 2y - 4x$

a) 

b)

Point	Slope	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	New Point
$(0,1)$	$2(1)-4(0) = 2$	$.1$	$\Delta y = 2(.1) = .2$	$(.1, 1.2)$
$(.1, 1.2)$	$2(1.2)-4(.1) = 2$	$.1$	$\Delta y = .2$	$\boxed{(.2, 1.4)}$

c) $y = 2x + b$
$y' = 2$

$\frac{dy}{dx} = 2y - 4x$

$2 = 2(2x+b) - 4x$

$2 = \cancel{4x} + 2b - \cancel{4x}$

$\boxed{b = 1}$

d) $(0,0)$

$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4$

$= 2(2y-4x) - 4$

$\frac{d^2y}{dx^2} = 4y - 8x - 4$ @ $(0,0)$

$= 4(0) - 8(0) - 4$

$= -4 \rightarrow$ $\boxed{\text{cc down} \therefore (0,0) \text{ is a MAX!}}$

#11 $(3,6)$ $f'(x) = \frac{1+e^x}{x^2} =$

a) $m = \frac{1+e^3}{3^2} = 2.343$

$y - 6 = 2.343(x-3)$

$y = 2.343x - 1.029$

$y = 2.343(3.1) - 1.029$

$\boxed{f(3.1) = y = 6.234}$

b)

Point	Slope	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	New Point
$(3,6)$	2.343	$.05$	$\Delta y = 2.343(.05) = .11715$	$(3.05, 6.11715)$
$(3.05, 6.11715)$	$\frac{1+e^{3.05}}{(3.05)^2} = 2.377$	$.05$	$y = 2.377(.05) = .11887$	$\boxed{(3.10, 6.236)}$

c) $\int_3^{3.1} f'(x)\,dx = .238$ ← in calc

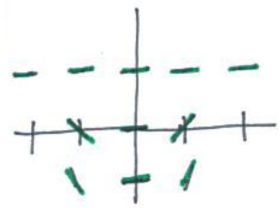
$f(x) \Big|_3^{3.1}$

$f(3.1) - f(3) = .238$

$f(3.1) - 6 = .238$

$\boxed{f(3.1) = 6.238}$

#12. $\frac{dy}{dx} = x(y-1)^2$

a) 

b) There is not a slope of zero @ $y=1$.

c) $\frac{dy}{dx} = x(y-1)^2 \quad f(0)=-1$

$\int \frac{dy}{(y-1)^2} = \int x\,dx$

$-(y-1)^{-1} = \frac{x^2}{2} + C$

$(y-1)^{-1} = -\frac{x^2}{2} + C$

$(-1-1)^{-1} = -\frac{0^2}{2} + C$

$-\frac{1}{2} = C$

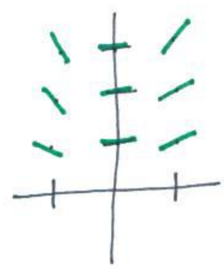
$\left((y-1)^{-1}\right)^{-1} = \left(\frac{-x^2}{2} - \frac{1}{2}\right)^{-1} \rightarrow \frac{x^2+1}{-2}$

$y-1 = \frac{-2}{x^2+1}$

$\boxed{y = \frac{-2}{x^2+1} + 1}$

d) Range: $-1 \le y < 1$

#13 $\frac{dy}{dx} = \frac{x \cdot y}{2}$

a) 

b)

Point	Slope	Δx	$\Delta y = \frac{dy}{dx} \cdot \Delta x$	New Point
$(0,3)$	$\frac{0 \cdot 3}{2} = 0$	$.1$	$\Delta y = 0(.1) = 0$	$(.1, 3)$
$(.1, 3)$	$\frac{(.1)(3)}{2} = .15$	$.1$	$\Delta y = .15(.1) = .015$	$\boxed{(.2, 3.015)}$

c) $\frac{dy}{dx} = \frac{x \cdot y}{2}$ @ $(0,3)$

$\int \frac{dy}{y} = \int \frac{x}{2}\,dx$

$\ln|y| = \frac{x^2}{4} + C$

$\ln 3 = \frac{0^2}{4} + C$

$C = \ln 3$

$e^{\ln|y|} = e^{\frac{x^2}{4}} + e^{\ln 3}$

$y = 3e^{x^2/4}$

$\boxed{f(.2) = 3e^{(.2)^2/4} = 3.030}$