Linearization & Newton's Method (Section 5.5)

*Linearization: If f is differentiable @ $x=a$, then the equation of the tangent line:

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

defines the linearization of $f(x)$ @ $x=a$, where "a" is the center of the approximation.

on HW: #1-8

ex: $f(x) = \sqrt{1+x}$ @ $x = \overset{a}{0}$; Approx: $\sqrt{1.02} \to x = .02$

$f(a) = f(0) = \sqrt{1+0} = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}(1) \to f'(0) = \frac{1}{2}(1+0)^{-1/2} = 1/2$

$L(x) = f(a) + f'(a)(x-a)$

$= 1 + 1/2(x-0)$

$$\boxed{L(x) = 1 + 1/2 x}$$

$L(.02) = 1 + 1/2(.02)$

$= 1 + 1/2(2/100)$

$\boxed{= 1.01}$ close!

actual $\sqrt{1.02} \approx 1.009950494$

$$\boxed{*\text{Error} = |\text{actual} - \text{approx}|}$$

$\text{Error} = |\sqrt{1.02} - 1.01| = \boxed{4.951 \times 10^{-5}}$

*Approximating Binomial Powers: for SMALL values of x:

$$(1+x)^k \approx 1 + k \cdot x$$

on HW #9-10

ex: $\sqrt{1+5x} = (1+\underbrace{5x}_{x})^{\overbrace{1/2}^{k}} \approx 1 + \frac{1}{2}(5x)$

$\boxed{\approx 1 + \frac{5}{2}x}$

ex: $\sqrt[3]{1-x} = (1-x)^{1/3}$

$\approx 1 + \frac{1}{3}(-x)$

$\boxed{\approx 1 - \frac{1}{3}x}$

* Approximating Roots: decide where to center the approximation & then linearize the function

HW #11-14

ex: $\sqrt{123}$ OR $\sqrt[3]{123}$ $\underline{a=123}$

closest perfect square: $121 = a$

$f(x) = \sqrt{x}$ $f(121) = 11$

$f'(x) = \frac{1}{2}x^{-1/2}$ $f'(121) = \frac{1}{2}\cdot\frac{1}{\sqrt{121}} = \frac{1}{22}$

$L(x) = 11 + \frac{1}{22}(x-121)$

$L(123) = 11 + \frac{1}{22}(123-121)$

$= 11 + \frac{1}{22}(2) = 11\tfrac{1}{11} \boxed{= 11.\overline{09}}$

actual $\sqrt{123} = 11.09053651$

Error $= \left|\sqrt{123} - 11.\overline{09}\right| = \boxed{3.726 \times 10^{-4}}$

closest perfect cube: $125 = a$

$f(x) = x^{1/3}$ $f(125) = 5$

$f'(x) = \frac{1}{3}x^{-2/3}$ $f'(125) = \frac{1}{3}\left(\frac{1}{\sqrt[3]{125}}\right)^2$

$= 1/75$

$L(123) = 5 + \frac{1}{75}(123-125)$

$= 5 - 2/75 \boxed{= 4.97\overline{3}}$

actual $\sqrt[3]{123} = 4.973189833$

Error $= \left|\sqrt[3]{123} - 4.97\overline{3}\right| = \boxed{1.435 \times 10^{-4}}$

* Differentials: Let $y = f(x)$ be differentiable function. The differential "dx" is an independent variable & $dy = f'(x) \cdot dx$

on HW #15-22

ex: $y = x^5 + 37x$; $x = 1$; $dx = .01$

$dy = 5x^4 dx + 37 dx$

$dy = (5x^4 + 37)dx$

$dy = [5(1)^4 + 37](.01) \boxed{= 0.42}$

$2 + y = 2y \rightarrow y = 2$

ex: $x + y = x \cdot y$; $x = 2$; $dx = .05$

$1dx + 1dy = x\,dy + y\,dx$

$dx(1-y) = dy(x-1)$

$dy = \left(\frac{1-y}{x-1}\right)dx = \left(\frac{1-2}{2-1}\right)(.05)$

$\boxed{dy = -.05}$

* The differential of f is df → basically the derivative w/ dx on the end.

on HW #23-26

ex: $f(x) = 3x^2 - 6$

$\boxed{df = 6x\,dx}$

on HW #27-30

ex: $f(x) = x^3 - x$; $a = 1$; $dx = 0.1$ Find the true change, est change, and error.

TRUE change: $\Delta f = f(a+dx) - f(a)$

$f(1) = (1)^3 - 1 = 0$

$f(1.1) = (1.1)^3 - 1.1 = .231$

$\Delta f = .231 - 0 \boxed{= .231}$

EST change: $df = f'(a)\,dx$

$f'(x) = 3x^2 - 1$

$df = [3(1)^2 - 1](.1)$

$\boxed{df = .2}$

ERROR:

$= |.231 - .2|$

$\boxed{= 0.031}$

* Newton's Method: A numerical technique of approximating a zero of a function.

STEPS: ① Guess a first approximation of $f(x) = 0$ (look @ the graph)

② Use:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The 1st approx finds the 2nd, the 2nd approx finds the 3rd...

③ Keep going until the approximation doesn't change.

IN CALCULATOR

$f(x) \rightarrow y_1$

$f'(x) \rightarrow y_2$

then use:

$$x_{n+1} = ANS - \frac{y_1(ANS)}{y_2(ANS)}$$

* Newton's method does not work if $f'(x_1) = 0$.

* Newton's method must converge to find a zero → if successive approx go back & forth between 2 values, it will not converge!

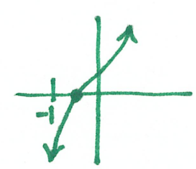
* starting value must be close to the zero you want to find.

on HW #53-56

ex: Use Newton's method to solve: $f(x) = x^3 + 3x + 1 = 0$.

$f(x) \rightarrow y_1$

$f'(x) = 3x^2 + 3 \rightarrow y_2$



zero is between $-1 \le x \le 0$

$x_1 = -.5$

$x_2 = -.\overline{3}$

$x_3 = -.3\overline{2}$

$x_4 = -.322185355$

$x_5 = -.3221853546$

$\boxed{x_6 = -.3221853546}$