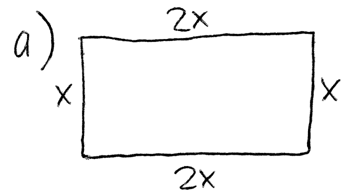
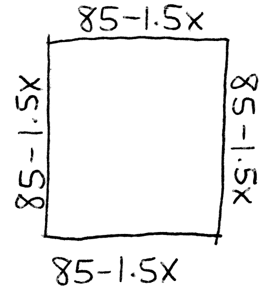
Optimization - Day 3: # 25 - 31

25. 340 yards total

a) 



$$\frac{340-6x}{4} = 85-1.5x$$

$P = 6x$ $\quad$ $P = 340-6x$

$2x \cdot x \geq 800$ $\quad$ $(85-1.5x)^2 \geq 100$

$2x^2 \geq 800$ $\quad$ $85-1.5x \geq 10$

$x^2 \geq 400$ $\quad$ $-1.5x \geq -75$

$x \geq 20$ $\quad$ $x \leq 50$

$$\boxed{20 \leq x \leq 50}$$

b) $A = 2x^2 + (85-1.5x)^2 = 2x^2 + 7225 - 255x + 2.25x^2$

$A = 4.25x^2 - 255x + 7225$

$A' = 8.5x - 255 = 0 \rightarrow 8.5x = 255 \rightarrow x = 30$

However, $A'' = 8.5 = + \rightarrow$ concave up, so $x=30$ is the minimum.

Since the critical point is a minimum, check the endpoints.

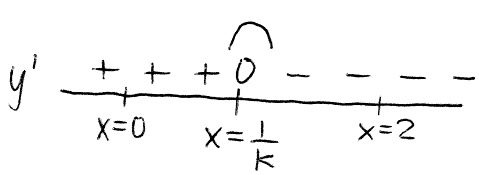
$A(20) = 3{,}825 \text{ yd}^2$

$A(50) = \boxed{5{,}100 \text{ yd}^2}$ ← Max Area

26. $y = xe^{-kx} = \dfrac{x}{e^{kx}}$

a) $y' = \dfrac{e^{kx}\cdot 1 - x\cdot e^{kx}\cdot k}{(e^{kx})^2} = \dfrac{\cancel{e^{kx}}(1-kx)}{\cancel{e^{kx}}\cdot e^{kx}} = \dfrac{1-kx}{e^{kx}}$ ← never $= 0$, so never DNE

$1-kx = 0 \rightarrow kx = 1 \rightarrow x = \frac{1}{k}$



$y'(0) = \dfrac{1-0}{e^0} = \dfrac{1}{1} = +$

$y'(2) = \dfrac{1-2k}{e^{2k}} = \dfrac{-}{+} = -$

Max bc sign of y' changes from + to −

$y(\frac{1}{k}) = \frac{1}{k}\cdot e^{-k\cdot(1/k)} = \frac{1}{k}\cdot e^{-1} = \frac{1}{ke} \rightarrow \boxed{\left(\frac{1}{k}, \frac{1}{ke}\right)}$

26. b) Abs. max at $x = \frac{1}{k}$ from part a

$$x = \frac{1}{k} \rightarrow k \cdot x = 1 \rightarrow k = \frac{1}{x}$$

$$\text{Max from a} = \frac{1}{ke} = \frac{1}{\left(\frac{1}{x}\right)e} = \frac{1}{\frac{e}{x}} = \boxed{\frac{x}{e}}$$

27. R = revenue, C = cost, P = profit

$R = 13x$ for $0 \le x \le 10$

$$C = \begin{cases} x^2+5x+7 & \text{for } 0 \le x \le 3 \\ x^2+5x+7+3(x-3) & \text{for } 3 < x \le 10 \end{cases}$$

Profit = Revenue - Cost

$$P = \begin{cases} 13x - x^2 - 5x - 7 & \text{for } 0 \le x \le 3 \\ 13x - x^2 - 5x - 7 - 3x + 9 & \text{for } 3 < x \le 10 \end{cases}$$

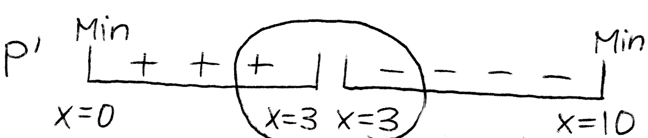
$$P = \begin{cases} -x^2+8x-7 & \text{for } 0 \le x \le 3 \\ -x^2+5x+2 & \text{for } 3 < x \le 10 \end{cases}$$

$$P' = \begin{cases} -2x+8 & \text{for } 0 \le x \le 3 \\ -2x+5 & \text{for } 3 < x \le 10 \end{cases}$$

$-2x+8 = 0 \rightarrow 2x = 8 \rightarrow x = 4$, but not in domain $0 \le x \le 3$

$-2x+5 = 0 \rightarrow 2x = 5 \rightarrow x = 2.5$, but not in domain $3 < x \le 10$

Neither of the theoretical critical points lie on their piece.



$P' = -2x+8$ $\qquad$ $P' = -2x+5$

$P'(1) = +$ $\qquad$ $P'(4) = -$

Max at $x=3$ bc sign of P' changes from + to -

$P(3) = -3^2+8(3)-7 = -9+24-7 = 24-16 = 8$

An output of $\boxed{3 \text{ tons}}$ maximizes profit at \$8 per ton

28. $x(t) = e^{-t}\sin t$ on $0 \le t \le 2\pi$

$x(t) = \dfrac{\sin t}{e^t}$

a) Find minimum position = farthest left

$$v(t) = \frac{e^t \cos t - \sin t\, e^t}{(e^t)^2} = \frac{\cancel{e^t}(\cos t - \sin t)}{\cancel{e^t} \cdot e^t} = \frac{\cos t - \sin t}{e^t} \quad \leftarrow \text{never} = 0$$

$\cos t - \sin t = 0 \rightarrow \cos t = \sin t$ at $t = \pi/4,\ 5\pi/4$

$v(t)$: $t=0$ | $+ + +$ | 0 at $t=\frac{\pi}{4}$ | $- - - -$ | 0 at $t=\frac{5\pi}{4}$ | $+ + +$ | $t=2\pi$ $\qquad \dfrac{\cos t - \sin t}{e^t}$

$v(\pi/6) = \dfrac{\sqrt{3}/2 - 1/2}{e^{\pi/6}} = \dfrac{+}{+} = + \qquad v(3\pi/2) = \dfrac{0 - -1}{e^{3\pi/2}} = \dfrac{+}{+} = +$

$v(\pi) = \dfrac{-1-0}{e^{\pi}} = \dfrac{-}{+} = -$

Farthest left at $\boxed{t = 5\pi/4}$ bc sign of v changes from $-$ to $+$

b) $A x''(t) + x'(t) + x(t) = 0$

$x(t) = \dfrac{\sin t}{e^t}, \quad x'(t) = \dfrac{\cos t - \sin t}{e^t}$

$$x''(t) = \frac{e^t(-\sin t - \cos t) - (\cos t - \sin t)e^t}{(e^t)^2} = \frac{\cancel{e^t}(\cancel{-\sin t} - \cos t - \cos t + \cancel{\sin t})}{\cancel{e^t} \cdot e^t}$$

$$x''(t) = \frac{-2\cos t}{e^t}$$

$$A\left(\frac{-2\cos t}{e^t}\right) + \frac{\cos t - \sin t}{e^t} + \frac{\sin t}{e^t} = 0$$

$-2A\cos t + \cos t \cancel{- \sin t} + \cancel{\sin t} = 0$

$-2A\cos t + \cos t = 0 \rightarrow \underbrace{1\cos t = 2A\cos t}_{\text{equivalent}} \rightarrow 2A = 1 \rightarrow \boxed{A = 1/2}$

29. $y = mx - \frac{e^{2m}}{1000}x^2$

a) Horizontal axis, so $y=0 \rightarrow mx - \frac{e^{2m}}{1000}x^2 = 0 \rightarrow mx = \frac{e^{2m}}{1000}x^2$

$m = \frac{e^{2m}}{1000}x \rightarrow x = \frac{1000m}{e^{2m}}$ (this equation is what we want to maximize)

$x' = \frac{e^{2m}\cdot 1000 - 1000m\cdot e^{2m}\cdot 2}{(e^{2m})^2} = \frac{\cancel{e^{2m}}(1000-2000m)}{\cancel{e^{2m}}\cdot e^{2m}} = \frac{1000-2000m}{e^{2m}}$ ← never $=$ 0

$1000 - 2000m = 0 \rightarrow 2000m = 1000 \rightarrow \boxed{m = \frac{1}{2}}$

x' $+++ \; 0 \; ---$ at $m=\frac{1}{2}$ Max ✓ $\qquad \frac{1000-2000m}{e^{2m}}$

$x'(0) = \frac{+}{+} = +$

$x'(1) = \frac{-}{+} = -$

b) $x = 100$, maximize y

$y = 100m - \frac{e^{2m}}{1000}\cdot 100^2 = 100m - 10e^{2m}$ (maximize this)

$y' = 100 - 10e^{2m}\cdot 2 = 100 - 20e^{2m} = 0$

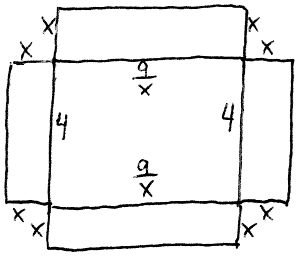
$20e^{2m} = 100 \rightarrow e^{2m} = 5 \rightarrow 2m\cdot \cancel{\ln e} = \ln 5 \rightarrow \boxed{m = \frac{\ln 5}{2}}$

y' $+++ \; 0 \; ---$ at $m = \frac{\ln 5}{2}$ Max ✓ $\qquad 100 - 20e^{2m}$

$y'(0) = 100 - 20e^0 = 100 - 20 = +$

$y'(\ln 5) = 100 - 20e^{2\ln 5} = 100 - 20\cancel{e^{\ln}}5^2 = 100 - 20\cdot 25 = -$

30.



$W = 4\ m,\ V = 36\ m^3$

$V = lwh$

$l \cdot 4 \cdot x = 36 \rightarrow 4xl = 36 \rightarrow l = \frac{36}{4x} = \frac{9}{x}$

Sides $= \$5/m^2$, Base $= \$10/m^2$

$$A = 4 \cdot \frac{9}{x} + 2 \cdot 4 \cdot x + 2 \cdot \frac{9}{x} \cdot x = \underbrace{\frac{36}{x}}_{\text{Base}} + \underbrace{8x + 18}_{\text{Sides}}$$

Cost = Area × Price

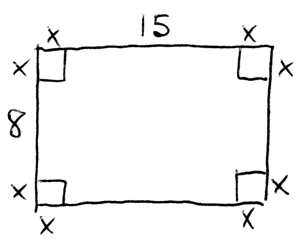
$$C = \frac{36}{x} \cdot 10 + 8x \cdot 5 + 18 \cdot 5 = \frac{360}{x} + 40x + 90 = 360x^{-1} + 40x + 90$$

$$C' = -360x^{-2} + 40 = \frac{-360}{x^2} + 40 = 0$$

$$40 = \frac{360}{x^2} \rightarrow 40x^2 = 360 \rightarrow x^2 = 9 \rightarrow x = 3$$

$$C(3) = \frac{360}{3} + 40 \cdot 3 + 90 = 120 + 120 + 90 = \boxed{\$330}$$

31.



$l = 15 - 2x$ $x < 7.5$

$w = 8 - 2x$ $x < 4$

$h = x$

$$V = (15-2x)(8-2x)x = (120 - 30x - 16x + 4x^2)x$$

$$V = 120x - 46x^2 + 4x^3$$

$$V' = 120 - 92x + 12x^2 = 0 \rightarrow \overset{a}{12}x^2 - \overset{b}{92}x + \overset{c}{120} = 0$$

$$x = \frac{92 \pm \sqrt{92^2 - 4(12)(120)}}{2(12)} = \frac{92 \pm \sqrt{2704}}{24}$$

6, but not in domain

$1.667 = \frac{5}{3}$

$$V(5/3) = 120(5/3) - 46(5/3)^2 + 4(5/3)^3 = \boxed{90.741\ in^3}$$