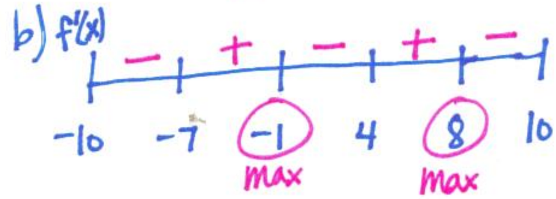
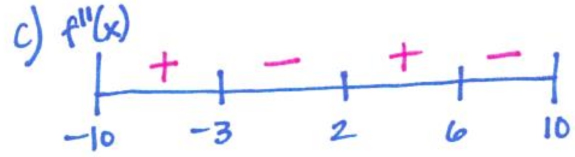
Optimization Packet

1. a) Horz tangent → $f'(x)=0$ ∴ $\boxed{x=-7,-1,4,8}$

b) 

$\boxed{\text{x=−1 & x=8 are rel. max(s) because f'(x) changes from (+) to (−).}}$

c) 

$\boxed{\text{ccdown: } (-3,2) \cup (6,10)}$

2. a) $f(x)=x^3-7x+6$ $\boxed{\text{zeros: } x=-3,1,2}$

b) $f'(x)=3x^2-7$

$f'(-1)=3(-1)^2-7=-4 \rightarrow$ pt$(-1,12)$ so... $\boxed{y-12=-4(x+1)}$

c) MVT: $f'(c)=\dfrac{f(3)-f(1)}{3-1}=\dfrac{12-0}{2}=6$

$f'(c)=3c^2-7=6$

$3c^2=13$

$\boxed{c=\pm\sqrt{13/3}\approx\pm 2.082}$

3. a) $P(x)=x^4+ax^3bx^2+cx+d$

symm to y-axis → even ∴ $\boxed{\begin{matrix}a=0\\c=0\end{matrix}}$ $P(0)=(0)^4+b(0)^2+d=1$

$\boxed{d=1}$

max: $(0,1)$

min: $(q,-3)$

$P(x)=x^4+bx^2+1$

$P'(x)=4x^3+2bx=0$

$2x(2x^2+b)=0$

$\downarrow$ $\downarrow$

$x=0$ $x=q$

$2q^2+b=0$

$b=-2q^2$

3a) $P(q) = q^4 - 2q^2(q^2) + 1 = 3$

$\qquad\qquad\qquad\qquad\qquad\quad -1 \quad -1$

$-q^4 = -4$

$\sqrt{q^4} = \sqrt{4}$

$q^2 = \pm 2$

$b = -2(\pm 2)$

$b = \pm 4 \rightarrow \boxed{b = -4}$ (because it gives a max @ (0,1))

$\boxed{P(x) = x^4 - 4x^2 + 1}$

b) $q^2 = \pm 2$

$\boxed{q = \pm\sqrt{2}}$

4a) $f(x) = \sqrt{1+6x}$ domain: $1 + 6x \geq 0$ $\boxed{\text{range: } [0, \infty)}$

$\boxed{x \geq 1/6}$

b) $f'(x) = \frac{1}{2}(1+6x)^{-1/2}(6)$

$f'(x) = \frac{3}{(1+6x)^{1/2}}$ $\quad f'(4) = \frac{3}{(1+6\cdot 4)^{1/2}} = \boxed{\frac{3}{5}}$

c) $(4,5)$ $\quad y - 5 = \frac{3}{5}(x-4)$

$y = \frac{3}{5}x + \frac{13}{5}$ $\quad \therefore \boxed{y\text{-int} = \frac{13}{5}}$

d) $1 = \frac{3}{(1+6x)^{1/2}}$

$(1+6x)^{1/2} = 3$

$1 + 6x = 9$

$x = 4/3 \rightarrow f(4/3) = \sqrt{1+6(4/3)} = 3$ $\quad \therefore \boxed{(4/3, 3)}$

5 a) $f(x) = \sin^3(x) + \sin^3|x| \rightarrow f(x) = \begin{cases} \sin^3 x - \sin^3 x = 0 \ ; \ x<0 \\ 2\sin^3 x \ ; \ x>0 \end{cases}$

$\boxed{f'(x) = 6\sin^2 x \cdot \cos x \ ; \ x>0}$

b) $\boxed{f'(x) = 0 ; \ x<0}$

c) $\lim_{x \to 0^-} f(x) = 0 \quad \lim_{x \to 0^+} f(x) = 2(\sin 0)^3 = 0$ ∴ f(x) is continuous!

d) $\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{0-0}{x} = 0$

$\lim_{x \to 0^+} \frac{2\sin^3 x - 0}{x-0} = \lim_{x \to 0^+} \frac{2\sin^3 x}{x} = \lim_{x \to 0^+} 2\sin^2 x \left(\cancelto{1}{\frac{\sin x}{x}}\right)$

$= 2(\sin 0)^2 = 0$

∴ since the left & right hand limits are equal, the derivative exists!

6a) $f(x) = x^3 - x^2 - 4x + 4$

$= x^2(x-1) - 4(x-1)$

$= (x^2-4)(x-1)$

$= (x+2)(x-2)(x-1)$ $\boxed{\text{zeros: } x = \pm 2, 1}$

b) $f'(x) = 3x^2 - 2x - 4$

$f'(-1) = 3(-1)^2 - 2(-1) - 4 = 1 \rightarrow$ pt $(-1, 6)$ $\boxed{y - 6 = 1(x+1)}$

c) Pt: (a, b) & $(0, -8)$

$m = \frac{b+8}{a}$

$3a^2 - 2a - 4 = \frac{b+8}{a}$

$3a^3 - 2a^2 - 4a - 8 = b$ $\quad \overset{b}{\downarrow}$ $f(a) = a^3 - a^2 - 4a + 4$

$3a^3 - 2a^2 - 4a - 8 = a^3 - a^2 - 4a + 4$

$2a^3 - a^2 - 12 = 0$

$a = 2 \rightarrow b = (2)^3 - (2)^2 - 4(2) + 4 = 0$

$\boxed{\therefore (a,b) = (2,0)}$

7a) $x^2 - xy + y^2 = 9$

$$2x - x\frac{dy}{dx} + y(-1) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y-x) = y-2x$$

$$\boxed{\frac{dy}{dx} = \frac{y-2x}{2y-x}}$$

b) vert tangent → m is undef. or denom = 0

$$2y - x = 0$$

$$x = 2y$$

$$(2y)^2 - (2y)y + y^2 = 9$$

$$3y^2 = 9$$

$$y = \pm\sqrt{3} \longrightarrow x = \pm 2\sqrt{3}$$

$$\boxed{(2\sqrt{3}, \sqrt{3}) \; \$ \; (-2\sqrt{3}, -\sqrt{3})}$$

c) $(0,3)$ $\quad \frac{dy}{dx} = \frac{3-2(0)}{2(3)-(0)} = \frac{1}{2}$

$$\frac{d^2y}{dx^2} = \frac{(2y-x)(\frac{dy}{dx}-2) - (y-2x)(2\frac{dy}{dx}-1)}{(2y-x)^2} = \frac{(2(3)-0)(\frac{1}{2}-2) - \cancelto{0}{(3-0)(2\cdot\frac{1}{2}-1)}}{(2(3)-0)^2}$$

$$= \frac{6(-\frac{3}{2})}{36} = \frac{-9}{36} = \boxed{\frac{-1}{4}}$$

8 a) $g^2(x) + h^2(x) = 1$

$g'(x) = h^2(x)$

$h(x) > 0$

$g(0) = 0$

Justify $h'(x) = -g(x) \cdot h(x)$

$2g(x)g'(x) + 2h(x)h'(x) = 0$

$2h(x)h'(x) = -2g(x)g'(x)$

$$h'(x) = \frac{-\cancel{2}g(x)g'(x)}{\cancel{2}h(x)}$$

$$h'(x) = \frac{-g(x)h^{\cancel{2}}(x)}{h\cancel{(x)}}$$

$$\boxed{h'(x) = -g(x)h(x) ✓}$$

b) Justify: Max @ $x=0$

$h'(0)=0$ **AND** $h''(0)<0$

$h'(0) = -\overset{0}{\cancel{g(0)}}h(0) = 0$ ✓

$h''(x) = -g(x)h'(x) + h(x) \cdot -g'(x)$

$h''(x) = -[g(x)h'(x) + h(x)g'(x)] \rightarrow h''(0) = -[\overset{0}{\cancel{g(0)h'(0)}} + h(0)g'(0)]$

$h''(0) = -h(0)g'(0)$

$= -h(0) \cdot h^2(0)$

$= \boxed{-}\underbrace{h^3(0)}_{\text{Always (+)}}$ $\quad \therefore h''(0) < 0$ ✓

c) Justify: Pt of Inft @ $x=0$

$g''(0)=0$ AND $g''(x)$ must change sign @ $x=0$

$g'(x) = h^2(x)$

$g''(x) = 2h(x)h'(x)$

$g''(0) = 2h(0) \cdot \overset{0}{\cancel{h'(0)}} = 0$ ✓

$g''(x) = 2h(x) \cdot h'(x)$

$= 2h(x)[-g(x)h(x)]$

$= \boxed{-2}\underbrace{h^2(x)}_{\text{Always (+)}}g(x)$ ↖ changes sign @ $x=0$

$\therefore$ $g''(x)$ changes sign @ $x=0$ ✓

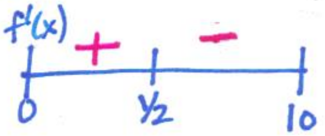
9 a) $f(x)=xe^{-2x} \qquad 0\le x\le 10$

$f'(x)=x\left[e^{-2x}(-2)\right]+e^{-2x}(1)$

$f'(x)=e^{-2x}(1-2x)=0$

$\downarrow \qquad \downarrow$

$\cancel{0} \qquad x=1/2$



inc: $[0,1/2)$

decr: $(1/2,10]$

b) $f(0)=0$

$f(1/2)=\frac{1}{2e}\approx 0.184$

$f(10)=\frac{10}{e^{20}}\approx 2.061\times10^{-8}$

$\therefore$ Absol Max → $(1/2, 1/2e)$

Absol Min → $(0,0)$

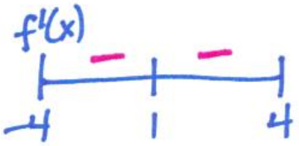
10 a) $f(x)=(x^2+1)e^{-x} \qquad -4\le x\le 4$

$f'(x)=(x^2+1)(-e^{-x})+e^{-x}(2x)$

$f'(x)=-e^{-x}(x^2-2x+1)=0$

$\downarrow \qquad \downarrow$

$\cancel{0} \qquad x=1$



$\therefore$ Absol Max must be @ the left end pt since $f(x)$ is always decreasing, so $x=-4$ is Absol Max

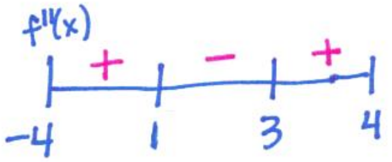
b) $f''(x)=-e^{-x}(2x-2)+(x^2-2x+1)(e^{-x})$

$=e^{-x}(x^2-4x+3)$

$=e^{-x}(x-3)(x-1)=0$

$\downarrow \qquad \downarrow \quad \downarrow$

$\cancel{0} \qquad x=3,1$



$\therefore$ $x=1$ & 3 are pts of infl. because $f''(x)$ changes sign @ these pts.

11 a) $f(-x) = f(x) \rightarrow$ symm to y-axis

$f(p) = 1$

$f'(p) = 5$

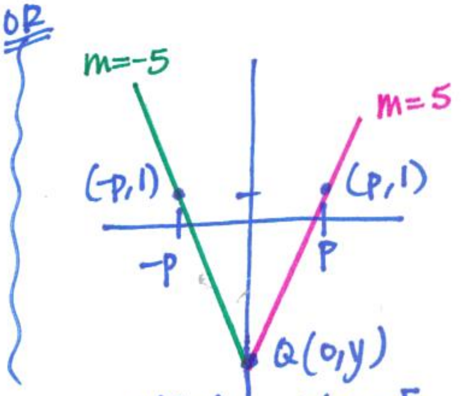
Find $f'(-p)$.

$-f'(-x) = f'(x)$

$-f'(-p) = f'(p)$

$$\boxed{f'(-p) = -5}$$

OR



$f'(-p)$ must be -5 by symmety.

b) Find $f'(0)$. By MVT: $f'(c) = \dfrac{f(p) - f(-p)}{p + {}^{-}p} = \dfrac{1-1}{2p} = 0$

$$\boxed{\therefore f'(0) = 0}$$

c) $Q(0, y)$

$y - 1 = 5(x-p)$ & $y - 1 = -5(x+p)$

$5x - 5p = -5x - 5p$

$10x = 0$

$x = 0$ ✓

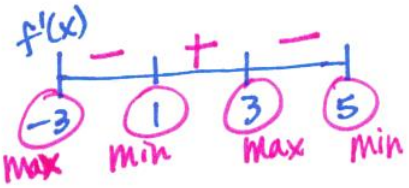
$y - 1 = -5(0+p)$

$y = 1 - 5p$

$$\boxed{\therefore Q(0, 1-5p)}$$

12 a) $f(x) = 2\ln(x^2+3) - x \quad -3 \le x \le 5$

$$f'(x) = 2\left(\frac{2x}{x^2+3}\right) - 1 \overset{\text{get CD}}{=} \frac{4x - x^2 - 3}{x^2+3} = \frac{-(x^2-4x+3)}{x^2+3} = \frac{-(x-3)(x-1)}{x^2+3} = 0 \quad \nearrow x = 3, 1$$



Rel Max: $x = -3$ end-pt then decr.

$x = 3$ $f'(x)$ changes from (+) to (-)

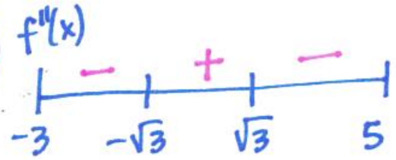
Rel Min: $x = 1$ $f'(x)$ changes from (-) to (+)

$x = 5$ decr. to end-pt

b) $$f''(x) = \frac{(x^2+3)(-2x+4) + (x^2-4x+3)(2x)}{(x^2+3)^2}$$

$$= \frac{\cancel{-2x^3} - \cancel{6x} + 4x^2 + 12 + \cancel{2x^3} - 8x^2 + \cancel{6x}}{(x^2+3)^2} = \frac{-4x^2+12}{(x^2+3)^2} = \frac{-4(x^2-3)}{(x^2+3)^2} \quad \nearrow x = \pm\sqrt{3}$$

12 b)



$\boxed{\therefore \text{Pt of Infl: } x = \pm\sqrt{3}}$

c) Absol Max: $\boxed{f(-3) = 2\ln 12 + 3}$

$f(3) = 2\ln 12 - 3$

13 a) $f(x) = \sin^2 x - \sin x \qquad 0 \le x \le {}^{3\pi}/_{2}$

$0 = \sin x\,(\sin x - 1)$

$\downarrow \qquad \downarrow$

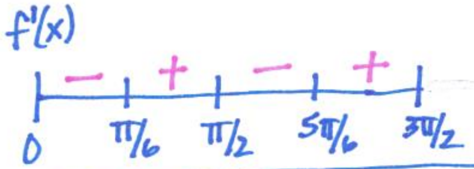
$\boxed{x = 0, \pi \qquad x = \pi/2}$

b) $f'(x) = 2\sin x\cos x - \cos x$

$0 = \cos x\,(2\sin x - 1)$

$\downarrow \qquad \downarrow$

$x = \pi/2, 3\pi/2 \qquad x = \pi/6, 5\pi/6$



$\boxed{\text{Incr: } (\pi/6, \pi/2) \cup (5\pi/6, 3\pi/2]}$

c) $f(\pi/6) = {}^{-}1/4 \qquad f(5\pi/6) = {}^{-}1/4$

$f(\pi/2) = 0 \qquad f(0) = 0$

$f(3\pi/2) = 2$

$\boxed{\begin{array}{l} \therefore \text{Absol Min} = {}^{-}1/4 \\ \text{Absol Max} = 2 \end{array}}$

14 a) $f(x) = \ln\left|\dfrac{x}{1+x^2}\right| \qquad$ Domain: $x \ne 0$

b) $\boxed{\text{Even:}}$ $f(-x) = f(x)$ ~~Odd: $f(-x) = -f(x)$~~

$f(-1) \overset{?}{=} f(1)$

$\ln|{}^{-}1/2| = \ln|1/2|$ ✓

c) $f(x) = \begin{cases} \ln\left(\dfrac{x}{1+x^2}\right) & x > 0 \\ \ln\left(\dfrac{-x}{1+x^2}\right) & x < 0 \end{cases}$

* Because of Symmetry, we can just use the (+) part of the function.

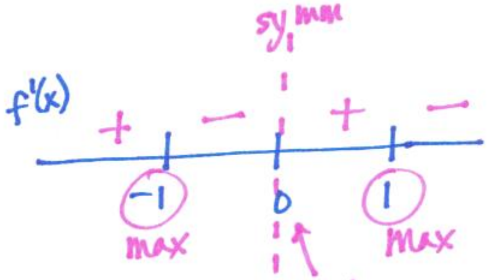
#14 c) $f'(x) = \dfrac{1}{\frac{x}{1+x^2}}\left[\dfrac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}\right]$; $x>0$

$= \dfrac{\cancel{1+x^2}}{x}\left[\dfrac{1+x^2-2x^2}{(1+x^2)^{\cancel{2}}}\right]$; $x>0$

from symmetry

$= \dfrac{1-x^2}{x(1+x^2)} = 0 \rightarrow x = 1, -1$

$x = 0$ ($f'(x)$ undef)



Not a min because $x=0$ is Not in the domain!

$\boxed{\begin{array}{l}\text{Rel Max: } x = 1, -1 \\ \text{Rel min: } \emptyset\end{array}}$

d) Range: $f(1) = \ln\left|\tfrac{1}{2}\right| = \ln \tfrac{1}{2}$

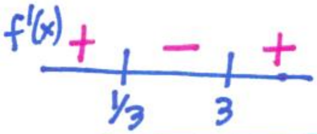
$f(-1) = \ln\left|-\tfrac{1}{2}\right| = \ln \tfrac{1}{2}$

$\boxed{\therefore \left(-\infty, \ln \tfrac{1}{2}\right]}$

15 a) $f(x)=x^3-5x^2+3x+k$

$f'(x)=3x^2-10x+3=0$

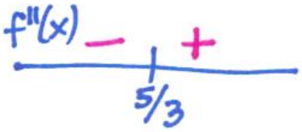
$x=1/3,\ 3$



$\boxed{\text{Incr: } (-\infty,1/3)\cup(3,\infty)}$

b) $f''(x)=6x-10=0$

$x=5/3$



$\boxed{\text{cc down: } (-\infty,5/3)}$

c) Mm: $f'(3)=0$

$f(3)=11=(3)^2-5(3)^2+3(3)+k$

$11=-9+k$

$\boxed{k=20}$

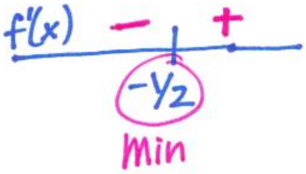
16 a) $f(x)=2x\cdot e^{2x}$ $\lim\limits_{x\to-\infty} f(x)=0$ $\lim\limits_{x\to\infty} f(x)=\infty$ or DNE

b) $f'(x)=2x\cdot 2e^{2x}+e^{2x}\cdot 2$

$0=2e^{2x}(2x+1)$

$\downarrow\qquad\downarrow$

$\cancel{0}\qquad x=-1/2$



$\boxed{\text{Min: } f(-1/2)=\frac{-1}{e}\approx 0.368}$

because $f'(x)$ changed from (−) to (+)

c) Range: $[-1/e,\infty)$

d) $f(x)=bxe^{bx}$

$f'(x)=bx\cdot be^{bx}+e^{bx}\cdot b$

$0=be^{bx}(bx+1)$

$\downarrow$

$\cancel{0}\qquad x=-1/b \rightarrow \boxed{f(-1/b)=\frac{-1}{e}}$

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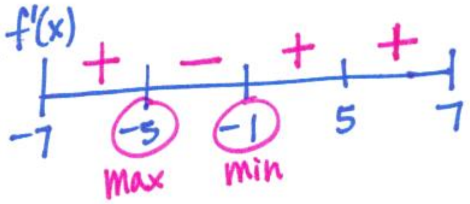


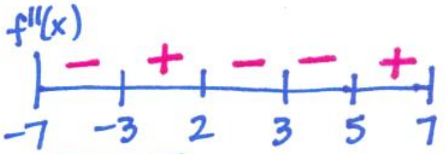
a) Rel Min: $x=-1 \rightarrow f'(x)$ changes from (−) to (+).

b) Rel Max: $x=-5 \rightarrow f'(x)$ changes from (+) to (−).

c) $f''(x)<0$

↓

$\boxed{[-7,-3) \cup (2,3) \cup (3,5)}$



18. $x^2+4y^2=7+3xy$

a) $2x+8y\frac{dy}{dx}=0+3x\frac{dy}{dx}+y(3)$

$\frac{dy}{dx}(8y-3x)=3y-2x$

$\boxed{\frac{dy}{dx}=\frac{3y-2x}{8y-3x}}$ ✓

b) Horz Tang: numerator $=0$

$3y-2(3)=0$

$3y=6$

$\boxed{y=2}$

$(3,2)=P$

$\frac{dy}{dx}=\frac{3(2)-2(3)}{8(2)-3(3)}=\frac{0}{7}=0$

c) $\frac{d^2y}{dx^2}=\frac{(8y-3x)(3\frac{dy}{dx}-2)-(3y-2x)(8\frac{dy}{dx}-3)}{(8y-3x)^2}$

$=\frac{[8(2)-3(3)][3(0)-2]-[\overset{0}{\cancel{3(2)-2(3)}}][8(0)-3]}{[8(2)-3(3)]^2}$

$=\frac{(16-9)(-2)}{49}=\frac{-14}{49}=\frac{-2}{7} \rightarrow$ $\boxed{y''<0 \text{ and } y'=0 \text{ @ } (3,2) \therefore (3,2) \text{ is a } \underline{\text{max}}!}$

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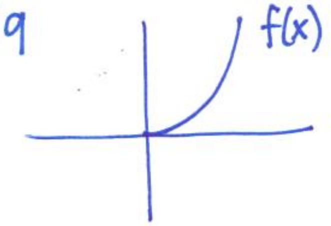


i) (0,0) and $f'(0)=0$

ii) (4,1) and $f'(4)=1$

iii) $0<x<4 \rightarrow f(x)$ is increasing

a) $f(x)=ax^2 \longrightarrow f(x)=\frac{1}{16}x^2$ $\quad f'(x)=\frac{1}{8}x$

$f(4)=a(4)^2=1$ $\qquad f'(4)=\frac{1}{8}(4)\overset{?}{=}1$

$16a=1$ $\qquad \frac{1}{2}\neq 1$ ∴ ii) does not work!

$a=1/16$ ✓

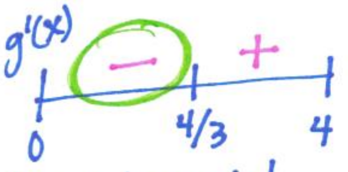
b) $g(x)=cx^3-\frac{x^2}{16} \longrightarrow g(x)=\frac{x^3}{32}-\frac{x^2}{16}$

$g(4)=c(4)^3-\frac{(4^2)}{16}=1$ $\qquad g'(x)=\frac{3}{32}x^2-\frac{1}{8}x\overset{?}{=}1$

$64c-1=1$ $\qquad g'(4)=\frac{3}{32}(4)^2-\frac{1}{8}(4)\overset{?}{=}1$

$c=1/32$ ✓ $\qquad =\frac{3}{2}-\frac{1}{2}=1$ ✓

c) $g'(x)=\frac{3}{32}x^2-\frac{1}{8}x=\underset{\substack{\downarrow\\ x=0}}{\frac{1}{8}x}\underset{\substack{\downarrow\\ x=4/3}}{\left(\frac{3}{4}x-1\right)}=0$



* g(x) is decr. from (0, 4/3), iii does not work!

d) $h(x)=\frac{x^n}{K} \longrightarrow h(x)=\frac{x^n}{4^n}=\left(\frac{x}{4}\right)^n$

$h(4)=\frac{4^n}{K}=1$ $\qquad h'(x)=n\left(\frac{x}{4}\right)^{n-1}\left(\frac{1}{4}\right)$

$K=4^n$ $\qquad h'(x)=\frac{n}{4}\left(\frac{x}{4}\right)^{n-1}$

$K=4^4$

$\boxed{K=256}$ $\qquad h'(4)=\frac{n}{4}\left(\frac{4}{4}\right)^{n-1}=1$

$\boxed{h(x)=\frac{x^4}{256}}$ $\qquad \frac{n}{4}=1$

$\boxed{n=4}$

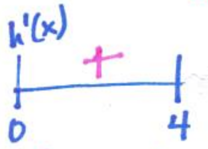
For i) $h(0)=\frac{0^4}{256}=0$ ✓

$h'(x)=\frac{4}{256}x^3$

$h'(0)=\frac{1}{64}(0)^3=0$ ✓

∴ i) works!

For iii)



∴ iii) works!

#20 $f(x) = K\sqrt{x} - \ln x \quad x>0$

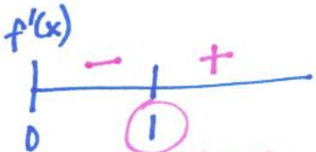
a) $f'(x) = K\cdot\frac{1}{2}x^{-1/2} - \frac{1}{x} = \boxed{\frac{K}{2}x^{-1/2} - x^{-1}}$

$f''(x) = \frac{K}{2}\cdot\frac{-1}{2}x^{-3/2} + x^{-2} = \boxed{\frac{-K}{4}x^{-3/2}+x^{-2}}$

b) $0 = \frac{K}{2}(1)^{-1/2} - (1)^{-1}$

$0 = \frac{K}{2} - 1$

$\boxed{K=2} \rightarrow f'(x) = x^{-1/2} - x^{-1}$



Rel min because f' changes from (−) to (+) @ x=1 when K=2.

c) $0 = \frac{-K}{4}x^{-3/2} + x^{-2} = x^{-2}\left(-\frac{K}{4}x^{1/2}+1\right)$

$\downarrow$ $x=0$ (DNE)

$\downarrow$ $\frac{K}{4}x^{1/2}=1$

$K=\frac{4}{\sqrt{x}}$

*"on the x-axis" means $y=0$

$0 = Kx^{1/2} - \ln x$

$\ln x = Kx^{1/2}$

$K = \frac{\ln x}{\sqrt{x}}$

$\frac{4}{\sqrt{x}} = \frac{\ln x}{\sqrt{x}}$

$4 = \ln x$

$\boxed{x=e^4}$

$K=\frac{4}{\sqrt{x}}$

$\boxed{K=\frac{4}{\sqrt{e^4}}=\frac{4}{e^2}}$

#21 $f(2)=5$; $f(5)=2$ $g(x)=f(f(x))$

a) $f'(c) = \frac{f(5)-f(2)}{5-2} = \frac{2-5}{3} = \frac{-3}{3} = -1$ (By MVT)

b) $g'(x) = f'(f(x))\cdot f'(x)$ so... $g'(2) = f'(f(2))\cdot f'(2)$ and $g'(5) = f'(f(5))\, f'(5)$

$= f'(5)\cdot f'(2)$ $\qquad = f'(2)\cdot f'(5)$

EQUAL!

c) $g''(x) = f'(f(x))\cdot \underset{\overset{\downarrow}{0}}{f''(x)} + f'(x)\cdot \underset{\overset{\downarrow}{0}}{f''(f(x))}\cdot f'(x)$

$g''(x)=0$ $\therefore$ No Critical Pts or Pts of Inflection!

d) $h(x) = f(x) - x \rightarrow h(2) = f(2)-2$ and $h(5) = f(5)-5$

$= 5-2$ $\qquad = 2-5$

$= 3$ $\qquad = -3$

By IVT, there must be an r such that $h(r)=0$

#22 $f(0)=2$; $f'(0)=-3$; $f''(0)=0$; $g'(x)=e^{-2x}\left[3f(x)+2f'(x)\right]$

a) Point: $(0,2)$

$m=-3$

$\boxed{\therefore\ y-2=-3(x-0)}$

b) Point of Inflection: @ $x=0$ ① $f''(0)=0$ $\downarrow$ given! ② $f''(x)$ changes sign @ $x=0$ $\downarrow$ Unknown!

$\boxed{\therefore \text{ Not Enough Info}}$

c) $g(0)=4$

$g'(0)=\overset{1}{\cancel{e^{0}}}\left[3f(0)+2f'(0)\right]$

$=3(2)+2(-3)=0$

$\boxed{\therefore\ y=4}$

d) $g''(x)=e^{-2x}\left[3f'(x)+2f''(x)\right]+\left[3f(x)+2f'(x)\right]\left(-2e^{-2x}\right)$

$=e^{-2x}\left[3f'(x)+2f''(x)-6f(x)-4f'(x)\right]$

$=e^{-2x}\left[-6f(x)-f'(x)+2f''(x)\right]$ ✓

*Max @ $x=0$: ① $g'(0)=0$ $\downarrow$ yes from part c. ✓ ② $g''(0)<0$

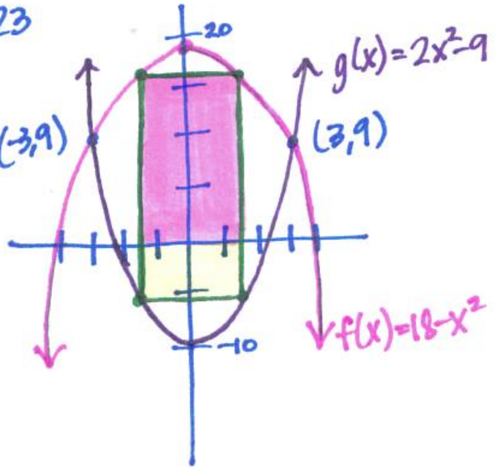
$g''(0)=\overset{1}{\cancel{e^{0}}}\left[-6f(0)-f'(0)+\overset{0}{\cancel{2f''(0)}}\right]$

$=-6(2)-(-3)$

$=-9<0$ ✓

$\boxed{\therefore \text{ yes, } x=0 \text{ is a max!}}$

#23



$A=\boxed{l\cdot w}+l\cdot w$

$A=2x\left(18-x^2\right)+2x\left[-\left(2x^2-9\right)\right]$

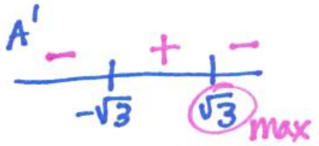
$A=36x-2x^3-4x^3+18x$

$A=-6x^3+54x$

$A'=-18x^2+54=0$

$-18\left(x^2-3\right)=0$

$x=\pm\sqrt{3}$



$A_{max}=-6\left(\sqrt{3}\right)^3+54\left(\sqrt{3}\right)$

$=-18\sqrt{3}+54\sqrt{3}$

$\boxed{=36\sqrt{3}}$

#24 $x(t) = (t-2)^3(t-6)$

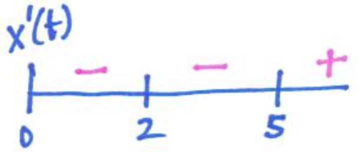
$x'(t) = (t-2)^3(1) + (t-6)[3(t-2)^2(1)]$

$= (t-2)^2[t-2+3(t-6)]$

$= (t-2)^2(4t-20)$

$= 4(t-2)^2(t-5) = 0$

$\downarrow$ $t=2$ $\quad$ $\downarrow$ $t=5$



a) Moving Right : $\boxed{(5,\infty)}$

b) At Rest : $\boxed{t=2,5}$

c) Change Direction: $\boxed{t=5}$

d) Farthest Left → Abs. Min

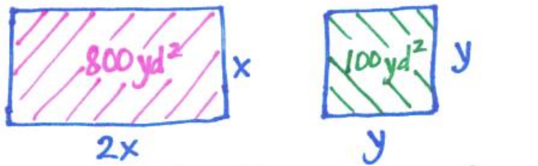
$x(5) = (5-2)^3(5-6)$

$= 27(-1)$

$= -27$

$\boxed{\therefore 27 \text{ units left}}$

#25



$A = x(2x) \geq 800$

$2x^2 \geq 800$

$x^2 \geq 400$

$x \geq 20$

$A = y^2 \geq 100$

$y \geq 10$

a) $P = 4y + 6x = 340$

$4y = 340 - 6x$

$y = 85 - \frac{3}{2}x \geq 10$

$-\frac{3}{2}x \geq -75$

$x \leq 50$

$\boxed{\therefore 20 \leq x \leq 50 \text{ so } x=50 \text{ max}, \ x=20 \text{ min}}$

b) $A = 2x^2 + y^2 = 2x^2 + \left(85 - \frac{3}{2}x\right)^2$

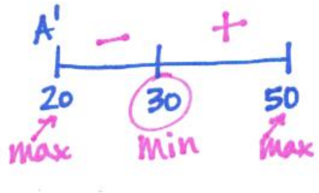
$A' = 4x + \cancel{2}\left(85 - \frac{3}{2}x\right)\left(-\frac{3}{\cancel{2}}\right)$

$= 4x - 255 + \frac{9}{2}x$

$0 = \frac{17}{2}x - 255$

$\downarrow$

$x = 30$



*when $x=20 \rightarrow y=55$

$x=50 \rightarrow y=10$

$A(20) = 2(20)^2 + (55)^2 = 3825$

$A(50) = 2(50)^2 + (10)^2 = 5100$

$\boxed{\therefore A_{max} = 5100 \text{ yd}^2}$

#26 $y = x \cdot e^{-kx}$

a) $y' = x\left[-ke^{-kx}\right] + e^{-kx}(1)$

$y(1/k) = \frac{1}{k}e^{-k(1/k)}$

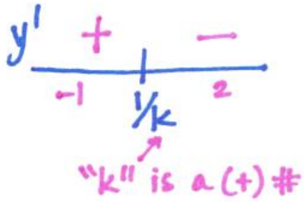
$= \frac{1}{k}e^{-1}$

$0 = e^{-kx}(-kx+1)$

$\downarrow$

$\cancel{0}$ $\quad kx = 1$

$x = 1/k$



$\boxed{\therefore \left(\frac{1}{k}, \frac{1}{ke}\right) \text{ is absolute max because } y' \text{ changes from (+) to (-)}}$

b) If $x = \frac{1}{k}$, then $k = \frac{1}{x}$

so if $y = \frac{1}{ke}$, then $\boxed{y = \frac{1}{\left(\frac{1}{x}\right)e} = \frac{x}{e} \text{ for } \underline{\underline{\text{any}}} \text{ "k" value.}}$

#27 Cost $\rightarrow$ $C(x) = \begin{cases} x^2+5x+7 \;; & 0 \le x \le 3 \\ \underbrace{x^2+5x+7+3(x-3)}_{x^2+8x-2} \;; & 3 < x \le 10 \end{cases}$

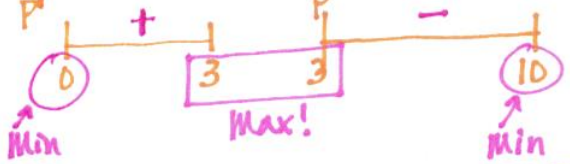
Profit $\rightarrow$ $P(x) = \begin{cases} 13x-(x^2+5x+7) = -x^2+8x-7 \;; & 0 \le x \le 3 \\ 13-(x^2+8x-2) = -x^2+5x+2 \;; & 3 < x \le 10 \end{cases}$

$P'(x) = \begin{cases} -2x+8 \;; & 0 \le x \le 3 \\ -2x+5 \;; & 3 < x \le 10 \end{cases}$

$8 = 2x$ $\qquad$ $2x = 5$

$\cancel{x=4}$ $\qquad$ $\cancel{x=5/2}$

Not in the Interval $\qquad$ Not in the interval



$P(3) = -(3)^2 + 8(3) - 7 = 8$

$\boxed{\therefore \text{Absol Max is 8 TONS}}$

#28 $x(t) = e^{-t}\sin t \quad 0 \le t \le 2\pi$

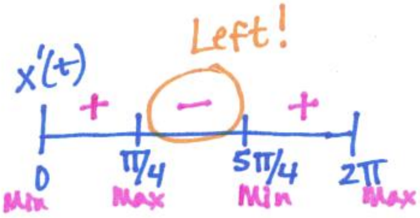
a) $x'(t) = e^{-t}(\cos t) + \sin t(-e^{-t})$

$0 = e^{-t}(\cos t - \sin t)$

$\downarrow \qquad\qquad \cos t = \sin t$

$\cancel{0} \qquad\qquad \downarrow$

$t = \pi/4, 5\pi/4$



Farthest Left → Abs. Min.

$x(0) = e^0 \sin(0) = 0$

$x(5\pi/4) = e^{-5\pi/4} \sin \frac{5\pi}{4} = (+)$

$\boxed{\therefore \text{Max left @ } t = \frac{5\pi}{4}}$

b) $x''(t) = e^{-t}[-\sin t - \cos t] + (\cos t - \sin t)(-e^{-t})$

$= -e^{-t}[\cancel{\sin t} + \cos t + \cos t \cancel{- \sin t}]$

$= -e^{-t}(2\cos t)$ OR $-2e^{-t}\cos t$

$0 = Ax''(t) + x'(t) + x(t)$

$0 = A(-2e^{-t}\cos t) + e^{-t}(\cos t - \sin t) + e^{-t}\sin t$

$0 = e^{-t}[-2A\cos t + \cos t \cancel{- \sin t} \cancel{+ \sin t}]$

$0 = e^{-t}\cos t(-2A + 1)$

$\downarrow \quad \downarrow \quad \downarrow$

$\cancel{0} \quad t = \pi/2, 3\pi/2 \quad \boxed{A = 1/2}$

#29 $y = mx - \frac{1}{1000} e^{2m} x^2$

a) Find m. Horz axis → $y=0$

$$mx - \frac{1}{1000} e^{2m} x^2 = 0$$

$$x\left(m - \frac{1}{1000} e^{2m} x\right) = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$x=0 \qquad m = \frac{1}{1000} e^{2m} x$$

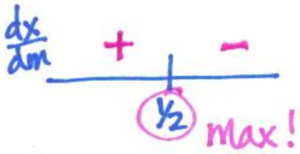
Maximize This! → $x = \frac{1000m}{e^{2m}}$ or $1000me^{-2m}$

$$\frac{dx}{dM} = (1000m)(-2e^{-2m}) + e^{-2m}(1000)$$

$$0 = 1000e^{-2m}(-2m+1)$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad m = 1/2$$



$\boxed{\therefore\ m = 1/2 \text{ is a max because } \frac{dx}{dm} \text{ changes from } (+) \text{ to } (-)}$

b) 100 ft from origin → $x = 100$

$$y = 100m - \frac{1}{1000} e^{2m} (100)^2$$

$$y = 100m - 10e^{2m}$$

$$\frac{dy}{dm} = 100 - 20e^{2m} = 0$$

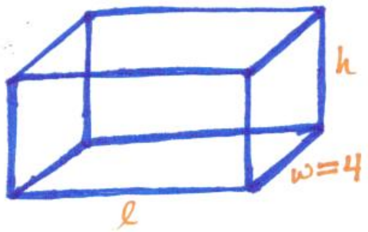
$$100 = 20e^{2m}$$

$$5 = e^{2m}$$

$$\ln 5 = 2m$$

$$\boxed{m = \frac{\ln 5}{2}}$$

#30



$SA = \overset{\$10}{l\cdot \overset{4}{w}} + \overset{\$5}{2\overset{4}{w}\cdot h} + \overset{\$5}{2l\cdot h}$

$SA = 4l + 8h + 2lh$

$SA = 4l + 8\left(\frac{9}{l}\right) + 2\cancel{l}\left(\frac{9}{\cancel{l}}\right)$

$SA = 4l + 72l^{-1} + 18$

$V = l\cdot w\cdot h = 4l\cdot h$

$36 = 4lh$

$9 = lh$

$h = \frac{9}{l}$

$h = \frac{9}{3} \rightarrow \boxed{h=3}$

$Cost = 40(3) + \frac{360}{3} + 90 = \boxed{\$330}$

Cost: $C(x) = 4l(10) + 72l^{-1}(5) + 18(5)$

$C(x) = 40l + 360l^{-1} + 90$

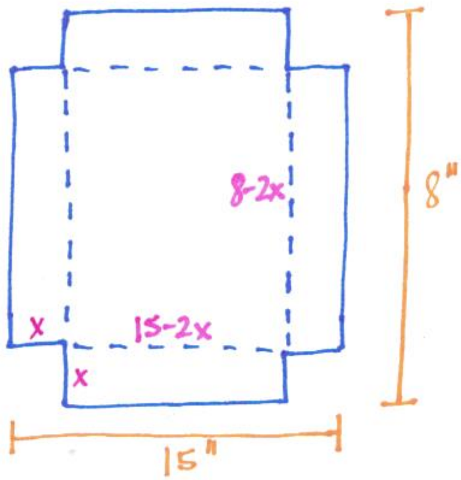
$c'(x) = 40 - 360l^{-2} = 0$

$40 = 360/l^2$

$l^2 = \frac{360}{40} \rightarrow l = \pm 3$

$\boxed{\therefore l = 3}$

#31



$x \geq 0$ $\quad$ $x \leq 4$

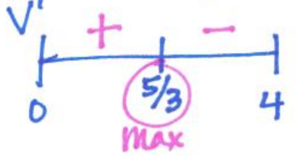
$V = x(15-2x)(8-2x)$ $\quad$ $\therefore 0 \leq x \leq 4$

$V = 120x - 46x^2 + 4x^3$

$V' = 12x^2 - 92x + 120 = 4(3x^2 - 23 + 30)$

$4(x-6)(3x-5) = 0$

$\cancel{x=6}$ $\quad$ $x = 5/3$



$V'' = 24x - 92$

$V''(5/3) = 24(5/3) - 92$

$= -52$ ∴ V is cc down @ $x = 5/3$ and this means it is a max!

$V_{max} = 120(5/3) - 46\left(\frac{5}{3}\right)^2 + 4\left(\frac{5}{3}\right)^3$

$\boxed{= \frac{2450}{27} \approx 90.741 \text{ in}^3}$