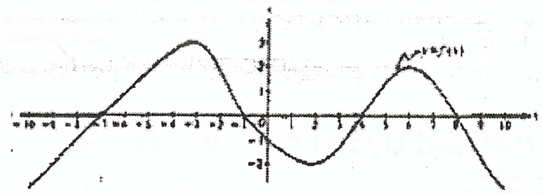
**AP Calculus Free Response – Applications of Derivatives (Optimization)**

***Show all work on a separate paper***

1.



The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10 \leq x \leq 10$.

a. For what values of x does the graph of f have a horizontal tangent?
b. For what values of x in the interval (-10, 10) does f have a relative maximum?
c. For what values of x is the graph of f concave downward?

2. Let f be the function given by $f(x) = x^3 - 7x + 6$.
   a. Find the zeros of f.
   b. Write an equation of the line tangent to the graph of f at $x = -1$.
   c. Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval [1,3].

3. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$. The graph of $y = P(x)$ is symmetric with respect to the y-axis, has a relative maximum at (0,1), and has an absolute minimum at (q, -3).
   a. Determine the values of a, b, c, and d, and using these values write an expression for P(x).
   b. Find all possible values for q.

4. Let f be a real-valued function defined by $f(x) = \sqrt{1+6x}$.
   a. Give the domain and range of f.
   b. Determine the slope of the line tangent to the graph of f at $x = 4$.
   c. Determine the y-intercept of the line tangent to the graph of f at $x = 4$.
   d. Give the coordinate of the point on the graph of f where the tangent line is parallel to $y = x + 12$.

5. Let f be the real-valued function defined by $f(x) = \sin^3(x) + \sin^3|x|$.

   a. Find $f'(x)$ for $x > 0$.

   b. Find $f'(x)$ for $x < 0$.

   c. Determine whether f(x) is continuous at $x = 0$. Justify your answer.

   d. Determine whether the derivative of f(x) exists at $x = 0$. Justify your answer.

6. Given the function f defined by $f(x) = x^3 - x^2 - 4x + 4$.

   a. Find the zeros of f.

   b. Write an equation of the line tangent to the graph of f at $x = -1$.

   c. The point (a,b) is on the graph of f and the line tangent to the graph at (a,b) passes through the point (0,-8) which is not on the graph of f. Find the value of a and b.

7. Given the curve $x^2 - xy + y^2 = 9$.

   a. Write a general expression for the slope of the curve.

   b. Find the coordinates of the points on the curve where the tangents are vertical.

   c. At the point (0, 3) find the rate of change in the slope of the curve with respect to x.

8. Let g and h be any two twice-differentiable functions that are defined for all real numbers and that satisfy the following properties for all x:

   (i) $(g(x))^2 + (h(x))^2 = 1$

   (ii) $g'(x) = (h(x))^2$

   (iii) $h(x) > 0$

   (iv) $g(0) = 0$

   a. Justify that $h'(x) = -g(x)h(x)$ for all x.

   b. Justify that h has a relative maximum at $x = 0$.

   c. Justify that the graph of g has a point of inflection at $x = 0$.

9. A function f is defined by $f(x) = x e^{-2x}$ with domain $0 \leq x \leq 10$.

   a. Find all values of x for which the graph of f is increasing and all values of x for which the graph decreasing.

   b. Give the x- and y-coordinates of all absolute maximum and minimum points on the graph of f. Justify your answers.

10. Let f be the function defined by $f(x) = (x^2 + 1)e^{-x}$ for $-4 \leq x \leq 4$.

   a. For what value of x does f reach its absolute maximum? Justify your answer.

   b. Find the x-coordinates of all points of inflection of f. Justify your answer.

11. For all real numbers x, f is a differentiable function such that f(-x) = f(x). Let f(p) = 1 and $f'(p) = 5$ for some p > 0.

a. Find $f'(p)$.

b. Find $f'(0)$.

c. If $l_1$ and $l_2$ are lines tangent to the graph of f at (-p,1) and (p,1), respectively, and if $l_1$ and $l_2$ intersect at point Q, find the x- and y-coordinates of Q in terms of p.

12. Let f be the function given by $f(x) = 2\ln(x^2 + 3) - x$ with domain $-3 \leq x \leq 5$.

a. Find the x-coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

b. Find the x-coordinate of each inflection point of f.

c. Find the absolute maximum value of f(x).

13. Give answers to this question in ***exact*** form:

Let f be the function defined by $f(x) = \sin^2 x - \sin(x)$ for $0 \leq x \leq \frac{3\pi}{2}$.

a. Find the x-intercepts of the graph of f.

b. Find the intervals on which f is increasing.

c. Find the absolute maximum value and the absolute minimum value of f. Justify your answer.

14. Let f be the function given by $f(x) = \ln\left|\frac{x}{1+x^2}\right|$.

a. Find the domain of f.

b. Determine whether f is an even function, an odd function, or neither. Justify your conclusion.

c. At what values of x does f have a relative maximum or a relative minimum? For each such x, use the first derivative test to determine whether f(x) is a relative maximum or a relative minimum.

d. Find the range of f.

15. Let f be the function given by $f(x) = x^3 - 5x^2 + 3x + k$, where k is a constant.

a. On what interval is f increasing?

b. On what intervals is the graph of f concave downward?

c. Find the value of k for which f has 11 as its relative minimum.

16. Let f be the function given by $f(x) = 2x e^{2x}$.

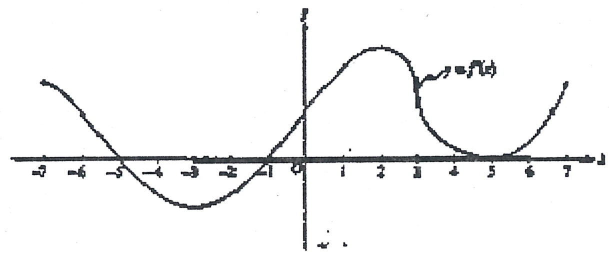
a. Find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.

b. Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

c. What is the range of f?

d. Consider the family of functions defined by $y = bx e^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of $bx e^{bx}$ is the same of all nonzero values of b.

17.



The figure above shows the graph of f $f'$, the derivative of the function f, for $-7 \leq x \leq 7$.

The graph of $f'$ has horizontal tangent lines at x = -3, x = 2, and x = 5, and a vertical tangent line at x = 3.

a. Find all values of x, for $-7 \leq x \leq 7$, at which f attains a relative minimum. Justify your answer.

b. Find all values of x, for $-7 \leq x \leq 7$, at which f attains a relative maximum. Justify your answer.

c. Find all values of x, for $-7 \leq x \leq 7$, at which $f''(x) < 0$.

d. At what value of x, for $-7 \leq x \leq 7$, does f attain its absolute maximum? Justify your answer.

18. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

a. Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

b. Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

c. Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part b. Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.