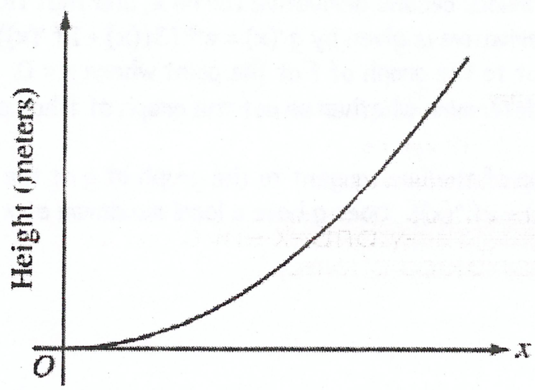
19.



The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.

(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.

(iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

20. Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

21. Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

(c) Show that if $f''(x) = 0$ for all x, then the graph of g does not have a point of inflection.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

22. Suppose that the function f has a continuous second derivative for all x, and that $f(0) = 2$, $f'(0) = -3$, and $f''(0) = 0$. Let g be a function whose derivative is given by $g'(x) = e^{-2x}(3f(x) + 2f'(x))$ for all x.

a. Write an equation of the line tangent to the graph of f at the point where $x = 0$.

b. Is there sufficient information to determine whether or not the graph of f has a point of inflection where $x = 0$? Explain your answer.

c. Given that $g(0) = 4$, write an equation of the line tangent to the graph of g at the point where $x = 0$.

d. Show that $g''(x) = e^{-2x}(-6f(x) - f'(x) + 2f''(x))$. Does g have a local maximum at $x = 0$? Justify your answer.

23. Find the area of the largest rectangle (with sides parallel to the coordinate axes) that can be inscribed in the region enclosed by the graphs of $f(x) = 18 - x^2$ and $g(x) = 2x^2 - 9$.

24. A particle starts at time $t = 0$ and moves on a number line so that its position at time t is given by $x(t) = (t - 2)^3 (t - 6)$.

a. When is the particle moving to the right?

b. When is the particle at rest?

c. When does the particle change direction?

d. What is the farthest to the left of the origin that the particle moves?

25. A man has 340 yards of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide and the other a square. The square field must contain at least 100 square yards and the rectangular one must contain at least 800 square yards.

a. If x is the width of the rectangular field, what are the maximum and minimum possible values of x?

b. What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.

26. a. Find the coordinate of the absolute maximum point for the curve $y = x e^{-kx}$ where k is a fixed positive number. Justify your answer.

b. Write an equation for the set of absolute maximum points for the curves $y = x e^{-kx}$ as k varies through positive values.

27. A manufacturer finds it costs him $x^2 + 5x + 7$ dollars to produce x tons of an item. At production levels above 3 tons, he must hire additional workers, and his costs increase by $3(x - 3)$ dollars on his total production. If the price he receives is \$13 per ton regardless of how much he manufactures and if he has a plant capacity of 10 tons, what level of output maximizes his profits?

28. A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time t at which the particle is farthest to the left. Justify your answer.

(b) Find the value of the constant A for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

29. A ball is thrown from the origin of a coordinate system. The equation of its path is $y = mx - \frac{1}{1000}e^{2m}x^2$, where m is positive and represents the slope of the path of the ball at the origin.

a. For what value of m will the ball strike the horizontal axis at the greatest distance from the origin? Justify your answer.

b. For what value of m will the ball strike at the greatest height on a vertical wall located 100 feet from the origin?

30. A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and volume is 36 cubic meters. If building the tank costs $10 per square meter for the base and $5 per square meter for the sides, what is the cost of the least expensive tank?

31. Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.