AP/IB Calculus

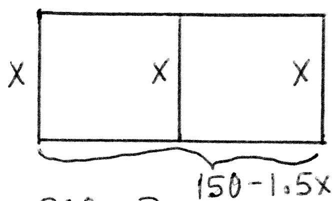
Name Key

Optimization Practice WS

Date________________ "A" Day Class Period____

Solve each optimization problem.

1) A rancher wants to construct two identical rectangular corrals using 300 ft of fencing. The rancher decides to build them adjacent to each other, so they share fencing on one side. What dimensions should the rancher use to construct each corral so that together, they will enclose the largest possible area?



$\frac{300-3x}{2} = 150 - 1.5x$

$A = x(150-1.5x) = 150x - 1.5x^2$

$A' = 150 - 3x$

$150 - 3x = 0 \rightarrow 3x = 150 \rightarrow x = 50$ (width)

$150 - 1.5x = 150 - 1.5(50) = 75$ (total length)

$\frac{75}{2} = 37.5$ ft (length of each corral)

Each corral: $\boxed{50 \text{ ft by } 37.5 \text{ ft}}$

2) A cryptography expert is deciphering a computer code. To do this, the expert needs to minimize the product of a positive rational number and a negative rational number, given that the positive number is exactly 8 greater than the negative number. What final product is the expert looking for?

Negative value $= x$, Positive value $= x+8$

$P = x(x+8) = x^2 + 8x$

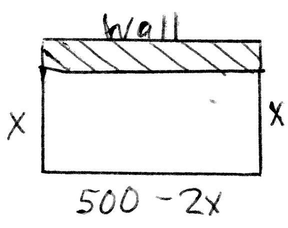
$P' = 2x + 8$

$2x + 8 = 0 \rightarrow 2x = -8 \rightarrow x = -4$ (negative value)

Positive value $= x + 8 = -4 + 8 = 4$

Final product $= -4 \cdot 4 = \boxed{-16}$

3) A farmer wants to construct a rectangular pigpen using 500 ft of fencing. The pen will be built next to an existing stone wall, so only three sides of fencing need to be constructed to enclose the pen. What dimensions should the farmer use to construct the pen with the largest possible area?



$A = x(500-2x) = 500x - 2x^2$

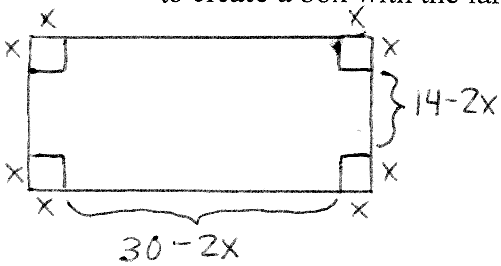
$A' = 500 - 4x$

$500 - 4x = 0 \rightarrow 4x = 500 \rightarrow x = 125$

$500 - 2x = 500 - 2(125) = 250$

Dimensions: $\boxed{125 \text{ ft by } 250 \text{ ft}}$

4) A supermarket employee wants to construct an open-top box from a 14 by 30 in piece of cardboard. To do this, the employee plans to cut out squares of equal size from the four corners so the four sides can be bent upwards. What size should the squares be in order to create a box with the largest possible volume?



$\ell = 30 - 2x$

$w = 14 - 2x$

$h = x$

$V = \ell w h$

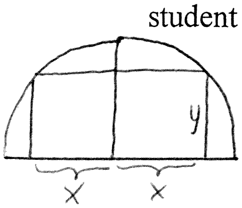
$$V = (30-2x)(14-2x)x = (420 - 60x - 28x + 4x^2)x$$
$$V = (420 - 88x + 4x^2)x = 4x^3 - 88x^2 + 420x$$
$$V' = 12x^2 - 176x + 420$$
$$\underset{a}{12}x^2 - \underset{b}{176}x + \underset{c}{420} = 0$$
$$x = \frac{176 \pm \sqrt{(-176)^2 - 4(12)(420)}}{2(12)}$$

$x = 11.667$ ← Impossible

$x = 3$

Size of squares: $\boxed{\text{3 in by 3 in}}$

5) A geometry student wants to draw a rectangle inscribed in a semicircle of radius 6. If one side must be on the semicircle's diameter, what is the area of the largest rectangle that the student can draw?



$x^2 + y^2 = r^2$

$x^2 + y^2 = 36$

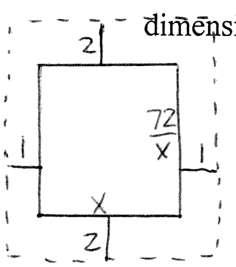
$y^2 = 36 - x^2$

$y = \sqrt{36 - x^2}$

$A = 2x \cdot y$

$$A = 2x\sqrt{36-x^2} = 2x(36-x^2)^{1/2}$$
$$A' = \cancel{2}x \cdot \tfrac{1}{\cancel{2}}(36-x^2)^{-1/2} \cdot -2x + (36-x^2)^{1/2} \cdot 2$$
$$A' = \frac{-2x^2}{\sqrt{36-x^2}} + \frac{2\sqrt{36-x^2}}{1} \cdot \frac{\sqrt{36-x^2}}{\sqrt{36-x^2}} = \frac{-2x^2 + 2(36-x^2)}{\sqrt{36-x^2}}$$
$$A' = \frac{-2x^2 + 72 - 2x^2}{\sqrt{36-x^2}} = \frac{72 - 4x^2}{\sqrt{36-x^2}} = 0 \quad \text{when } 72 - 4x^2 = 0$$
$$4x^2 = 72 \to x^2 = 18 \to x = \sqrt{18}$$
$$A = 2\sqrt{18} \cdot \sqrt{36 - \sqrt{18}^2} = 2\sqrt{18} \cdot \sqrt{18} = 2 \cdot 18 = \boxed{36}$$

6) A graphic designer is asked to create a movie poster with a 72 in² photo surrounded by a 2 in border at the top and bottom and a 1 in border on each side. What overall dimensions for the poster should the designer choose to use the least amount of paper?



$b = x + 2$

$h = \frac{72}{x} + 4$

$A = bh$

$$A = (x+2)\left(\frac{72}{x} + 4\right) = 72 + 4x + \frac{144}{x} + 8 = 144x^{-1} + 4x + 80$$
$$A' = -144x^{-2} + 4 = \frac{-144}{x^2} + 4$$
$$\frac{-144}{x^2} + 4 = 0 \to \frac{144}{x^2} = 4 \to 4x^2 = 144 \to x^2 = 36 \to x = 6$$

Poster: 6 in by 12 in

Overall dimensions: $b = x + 2 = 6 + 2 = 8$ in

$h = \frac{72}{x} + 4 = \frac{72}{6} + 4 = 12 + 4 = 16$ in

$\boxed{\text{8 in by 16 in}}$

7) Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 864 ft³ of water. What dimensions should they use to create an acceptable aquarium with the least amount of glass?

$V = x^2 h \to \quad x^2 h = 864 \to \quad \frac{864}{x^2} = h$

$S = x^2 + 4xh = x^2 + 4x \cdot \frac{864}{x^2} = x^2 + \frac{3456}{x} = x^2 + 3456x^{-1}$

$S' = 2x - 3456x^{-2}$

$2x - \frac{3456}{x^2} = 0 \to \quad 2x = \frac{3456}{x^2} \to \quad 2x^3 = 3456 \to x^3 = 1728 \to \quad x = 12$

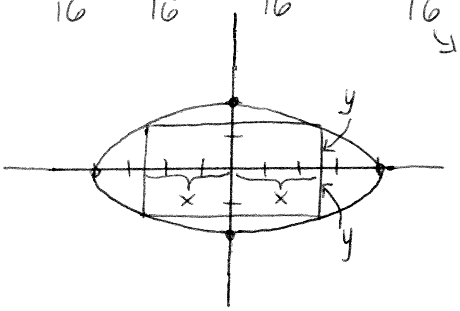
$h = \frac{864}{x^2} = \frac{864}{12^2} = 6$

Dimensions : $\boxed{12 \text{ ft by } 12 \text{ ft by } 6 \text{ ft}}$

8) A geometry student wants to draw a rectangle inscribed in the ellipse $x^2 + 4y^2 = 16$. What is the area of the largest rectangle that the student can draw?

$\frac{x^2}{16} + \frac{4y^2}{16} = \frac{16}{16} \to \quad \frac{x^2}{16} + \frac{y^2}{4} = 1$ $a = 4$ $b = 2$

bc general ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$b = 2x$

$h = 2y$

$A = bh = 4xy$

$x^2 + 4y^2 = 16$

$4y^2 = 16 - x^2$

$y^2 = 4 - \frac{1}{4}x^2 \to \quad y = \sqrt{4 - \frac{1}{4}x^2}$

$A = 4xy = 4x\sqrt{4 - \frac{1}{4}x^2} = 4x\left(4 - \frac{1}{4}x^2\right)^{\frac{1}{2}}$

$A' = \cancel{4}x \cdot \frac{1}{\cancel{2}}\left(4 - \frac{1}{4}x^2\right)^{-1/2} \cdot -\frac{1}{\cancel{2}}x + \left(4 - \frac{1}{4}x^2\right)^{1/2} \cdot 4$

$A' = \frac{-x^2}{\sqrt{4 - \frac{1}{4}x^2}} + \frac{4\sqrt{4 - \frac{1}{4}x^2}}{1} \cdot \frac{\sqrt{4 - \frac{1}{4}x^2}}{\sqrt{4 - \frac{1}{4}x^2}} = \frac{-x^2 + 4\left(4 - \frac{1}{4}x^2\right)}{\sqrt{4 - \frac{1}{4}x^2}}$

$A' = \frac{-x^2 + 16 - x^2}{\sqrt{4 - \frac{1}{4}x^2}} = \frac{-2x^2 + 16}{\sqrt{4 - \frac{1}{4}x^2}}$

$A' = 0$ if $-2x^2 + 16 = 0 \to \quad 2x^2 = 16 \to \quad x^2 = 8 \to \quad x = \sqrt{8}$

$A(\sqrt{8}) = 4 \cdot \sqrt{8} \cdot \sqrt{4 - \frac{1}{4}\sqrt{8}^2} = 4 \cdot \sqrt{8} \cdot \sqrt{4 - 2} = 4 \cdot \sqrt{8} \cdot \sqrt{2} = 4 \cdot \sqrt{16} = 4 \cdot 4 = \boxed{16}$

9) Which point on the graph of $y = \sqrt{x}$ is closest to the point $(3, 0)$?

$(3,0)$ and $(x, \sqrt{x})$ so $\Delta y = \sqrt{x}$ and $\Delta x = x - 3$

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x-3)^2 + \sqrt{x}^2} = \sqrt{x^2 - 6x + 9 + x} = \sqrt{x^2 - 5x + 9}$$

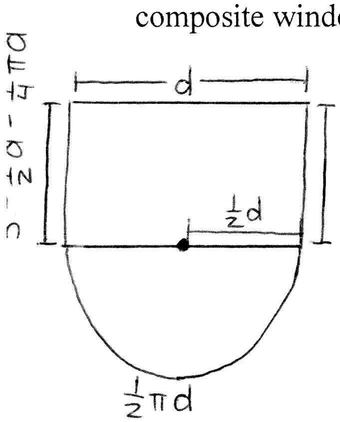
$$D = (x^2 - 5x + 9)^{1/2}$$

$$D' = \tfrac{1}{2}(x^2 - 5x + 9)^{-1/2}(2x - 5) = \frac{2x - 5}{2\sqrt{x^2 - 5x + 9}}$$

$D' = 0$ if $2x - 5 = 0 \rightarrow 2x = 5 \rightarrow x = 5/2$

Point : $\boxed{\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)}$

10) An architect is designing a composite window by attaching a semicircular window on top of a rectangular window, so the diameter of the top window is equal to and aligned with the width of the bottom window. If the architect wants the perimeter of the composite window to be 10 ft, what dimensions should the bottom window be in order to create the composite window with the largest area?



$$\frac{10 - d - \frac{1}{2}\pi d}{2} = 5 - \tfrac{1}{2}d - \tfrac{1}{4}\pi d$$

$5 - \frac{1}{2}d - \frac{1}{4}\pi d$

$$A = d\left(5 - \tfrac{1}{2}d - \tfrac{1}{4}\pi d\right) + \tfrac{1}{2}\pi\left(\tfrac{1}{2}d\right)^2$$

$$A = 5d - \tfrac{1}{2}d^2 - \tfrac{1}{4}\pi d^2 + \tfrac{1}{8}\pi d^2 = 5d - \tfrac{1}{2}d^2 - \tfrac{1}{8}\pi d^2$$

$$A' = 5 - d - \tfrac{1}{4}\pi d$$

$$5 - d - \tfrac{1}{4}\pi d = 0 \rightarrow 5 = d + \tfrac{1}{4}\pi d \rightarrow 5 = d\left(1 + \tfrac{1}{4}\pi\right)$$

$$d = \frac{5}{1 + \pi/4} \approx 2.800 \text{ ft}$$

$$5 - \tfrac{1}{2}d - \tfrac{1}{4}\pi d \approx 1.400 \text{ ft}$$

Dimensions : $\boxed{2.800 \text{ ft by } 1.400 \text{ ft}}$

Answers to Optimization Practice WS

1) $\frac{75}{2}$ ft (non-adjacent sides) by 50 ft (adjacent sides)

2) −16

3) 125 ft (perpendicular to wall) by 250 ft (parallel to wall)

4) 3 in

5) 36

6) 8 in wide by 16 in tall

7) 12 ft by 12 ft by 6 ft tall

8) 16

9) $\left(\frac{5}{2}, \frac{\sqrt{10}}{2}\right)$

10) $\frac{20}{4+\pi}$ ft (width) by $\frac{10}{4+\pi}$ ft (height)