Optimization - Day 1: #1-12

1. a) Horizontal tangent when $f'(x)=0$: $\boxed{x=-7,-1,4,8}$

   b) Rel. max when $f'(x)$ changes from + to − : $\boxed{x=-1,8}$

   c) Concave down when $f'(x)$ is decreasing: $\boxed{(-3,2)\cup(6,10)}$

2. $f(x)=x^3-7x+6$

   a) $x^3-7x+6=0$

   $1^3-7\cdot1+6 = 1-7+6=0$ ✓ $\rightarrow \boxed{x=1}$

   $$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

   $x^2+x-6=0$

   $(x+3)(x-2)=0 \rightarrow \boxed{x=-3,2}$

   b) $f(-1)=(-1)^3-7(-1)+6=-1+7+6=12 \rightarrow (-1,12)$

   $f'(x)=3x^2-7$

   $f'(-1)=3(-1)^2-7=3-7=-4$

   $y-12=-4(x+1)$

   $y-12=-4x-4$

   $\boxed{y=-4x+8}$

   c) $f(1)=1-7+6=0 \rightarrow (1,0)$

   $f(3)=27-21+6=12 \rightarrow (3,12)$

   Avg. ROC $=\dfrac{f(3)-f(1)}{3-1}=\dfrac{12-0}{3-1}=\dfrac{12}{2}=6$

   $3x^2-7=6 \rightarrow 3x^2=13 \rightarrow x^2=\frac{13}{3} \rightarrow \boxed{x=\sqrt{\frac{13}{3}}\approx 2.082}$

3. $P(x)=x^4+ax^3+bx^2+cx+d$

   a) Using $(0,1) \rightarrow 0+0+0+0+d=1 \rightarrow d=1$

   $P(x)=x^4+ax^3+bx^2+cx+1$

   Even: $f(-x)=f(x)$

   $(-x)^4+a(-x)^3+b(-x^2)+c(-x)+\cancel{1}=x^4+ax^3+bx^2+cx+\cancel{1}$

   $\cancel{x^4}-ax^3+\cancel{bx^2}-cx=\cancel{x^4}+ax^3+\cancel{bx^2}+cx$

   $-ax^3-cx=ax^3+cx$

   $-a=a$ Only true if $a=0$

   $-c=c$ Only true if $c=0$

   $\rightarrow$ $P(x)=x^4+bx^2+1$

   $P'(x)=4x^3+2bx=0$ when $P(x)$ has min or max

   $2x(2x^2+b)=0$

   $2x=0 \rightarrow x=0 \rightarrow$ Rel max

   $2x^2+b=0 \rightarrow x^2=\frac{-b}{2} \rightarrow$ Rel. min

3. a) continued

Rel. min at $x^2 = \frac{-b}{2}$, point $(q, -3)$

$$x^4 + bx^2 + 1 = -3$$
$$\left(\frac{-b}{2}\right)^2 + b\left(\frac{-b}{2}\right) + 1 = -3$$
$$\frac{b^2}{4} - \frac{b^2}{2} = -4$$
$$-\frac{b^2}{4} = -4$$

$$\rightarrow -b^2 = -16$$
$$b^2 = 16$$
$$b = \pm 4$$

If $b = 4$, $P(x) = x^4 + 4x^2 + 1$

$P'(x) = 4x^3 + 8x = 0$ at $x = 0$

$P''(x) = 12x^2 + 8 = 8$ at $x = 0 \rightarrow$ concave up $\rightarrow$ min at $x = 0$, but need max at $x = 0$

If $b = -4$, $P(x) = x^4 - 4x^2 + 1$

$P'(x) = 4x^3 - 8x = 0$ at $x = 0$

$P''(x) = 12x^2 - 8 = -8$ at $x = 0 \rightarrow$ concave down $\rightarrow$ max at $x = 0$ ✓

$\boxed{P(x) = x^4 - 4x^2 + 1}$

b) $P'(x) = 4x^3 - 8x = 0$

$4x(x^2 - 2) = 0$

$4x = 0 \quad x^2 - 2 = 0$

$x = 0 \quad x = \pm\sqrt{2}$

$P'(x)$ $\quad - - \underset{x=-\sqrt{2}}{\overset{\cup}{0}} + + + \underset{x=0}{\overset{\cap}{0}} - - - \underset{x=\sqrt{2}}{\overset{\cup}{0}} + + +$ $\quad 4x(x^2-2)$

$P'(-2) = -8(2) = -$

$P'(-1) = -4(-1) = +$

$P'(1) = 4(-1) = -$

$P'(2) = 8(2) = +$

Rel max: $x = 0$

$\boxed{\text{Rel min: } x = \pm\sqrt{2}}$

bc sign of $P'(x)$ changes from $-$ to $+$

4. $f(x) = \sqrt{1 + 6x}$

a) $1 + 6x \geq 0$

$6x \geq -1$

$x \geq -1/6$

$\boxed{D: [-1/6, \infty)}$

$\sqrt{1 + 6(-1/6)}$

$\sqrt{1 + -1}$

$\sqrt{0} = 0$

$\boxed{R: [0, \infty)}$

b) $f(x) = (1 + 6x)^{1/2}$

$f'(x) = \frac{1}{2}(1 + 6x)^{-1/2} \cdot 6 = \frac{3}{\sqrt{1 + 6x}}$ at $x = 4$: $\frac{3}{\sqrt{1 + 24}} = \boxed{\frac{3}{5}}$

4. c) $f'(4) = \frac{3}{5}$ (from part b)

$f(4) = \sqrt{1+6\cdot 4} = \sqrt{25} = 5 \rightarrow (4,5)$

$y - 5 = \frac{3}{5}(x-4) \rightarrow y = \frac{3}{5}x + \frac{13}{5} \rightarrow$ y-int: $\boxed{(0, 13/5)}$

d) Parallel to $y = x+12 \rightarrow$ slope $= 1$

$\frac{3}{\sqrt{1+6x}} = 1 \rightarrow 3 = \sqrt{1+6x} \rightarrow 1+6x = 9 \rightarrow 6x = 8 \rightarrow x = 4/3$

$f(4/3) = \sqrt{1+6(4/3)} = \sqrt{1+8} = \sqrt{9} = 3 \rightarrow \boxed{(4/3, 3)}$

5. $f(x) = \sin^3 x + \sin^3 |x|$

a) $f(x) = \sin^3 x + \sin^3 x = 2\sin^3 x$ when $x > 0$

$f'(x) = \boxed{6\sin^2 x \cos x}$

b) When $x < 0$, $|x| = -x$

$f(x) = \sin^3 x + \sin^3(-x)$

↑ sine is odd, so $\sin^3(-x) = -\sin^3 x$

$f(x) = \sin^3 x - \sin^3 x = \boxed{0} = f'(x)$

c) As $x \rightarrow 0^-$, $f(x) = 0$, so $f(0) = 0$

As $x \rightarrow 0^+$, $f(x) = 2(\sin x)^3$, so $f(0) = 2(\sin 0)^3 = 2 \cdot 0^3 = 0$

Yes, $f(x)$ is $\boxed{\text{continuous}}$ at $x=0$ bc the left & right hand limits agree.

d) $\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \frac{0-0}{x} = \frac{0}{x} = 0$

$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \frac{2\sin^3 x - 0}{x - 0} = \lim_{x \rightarrow 0^+} \frac{2\sin^3 x}{x} = \frac{0}{0} \rightarrow$ indeterminate form

$\lim_{x \rightarrow 0^+} \frac{2\sin^2 x}{1} \cdot \frac{\sin x}{x} = \lim_{x \rightarrow 0^+} 2\sin^2 x \cdot 1 = 2(\sin 0)^2 = 2 \cdot 0^2 = 0$

Slope as $x \rightarrow 0^-$ = Slope as $x \rightarrow 0^+ = 0$, so $f(x)$ is $\boxed{\text{differentiable}}$ at $x = 0$.

6. $f(x) = x^3 - x^2 - 4x + 4$

a) $x^3 - x^2 - 4x + 4 = 0$

At $\boxed{x=1}$: $1 - 1 - 4 + 4 = 0$ ✓

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -4 & 4 \\ & & 1 & 0 & -4 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

$x^2 - 4 = 0$

$\boxed{x = \pm 2}$

b) $f(-1) = -1 - 1 + 4 + 4 = 6 \rightarrow (-1, 6)$

$f'(x) = 3x^2 - 2x - 4$

$f'(-1) = 3 + 2 - 4 = 1$

$y - 6 = 1(x+1)$

$y - 6 = x + 1$

$\boxed{y = x + 7}$

c) (a, b) $(0, -8)$

$m = \frac{\Delta y}{\Delta x} = \frac{b - -8}{a - 0} = \frac{b+8}{a}$

$f'(x) = 3x^2 - 2x - 4$

$3x^2 - 2x - 4 = \frac{b+8}{a}$

$3a^2 - 2a - 4 = \frac{b+8}{a}$ (derivative at $x = a$)

$3a^3 - 2a^2 - 4a = b + 8$

$b = 3a^3 - 2a^2 - 4a - 8$

Use (a, b) in $f(x) = x^3 - x^2 - 4x + 4 \rightarrow a^3 - a^2 - 4a + 4 = b$

Both equal to b, so set expressions equal to each other

$3a^3 - 2a^2 \cancel{-4a} - 8 = a^3 - a^2 \cancel{-4a} + 4$

$2a^3 - a^2 - 12 = 0$

Graph on calculator: $x = 2 \rightarrow \boxed{a = 2}$

$b = a^3 - a^2 - 4a + 4$

$b = 8 - 4 - 8 + 4 = 0 \rightarrow \boxed{b = 0}$

7. $x^2 - xy + y^2 = 9$

a) $2x - x\frac{dy}{dx} + y(-1) + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \boxed{\frac{y-2x}{2y-x}}$

b) Vertical tangent → denominator = 0

$2y - x = 0 \rightarrow x = 2y$

$(2y)^2 - (2y)y + y^2 = 9 \rightarrow 4y^2 - 2y^2 + y^2 = 9 \rightarrow 3y^2 = 9 \rightarrow y^2 = 3 \rightarrow y = \pm\sqrt{3}$

$x = 2(\pm\sqrt{3}) = \pm 2\sqrt{3}$

$\boxed{(2\sqrt{3}, \sqrt{3})}$ and $\boxed{(-2\sqrt{3}, -\sqrt{3})}$

c) $\frac{dy}{dx}$ at $(0,3) = \frac{3-2(0)}{2(3)-0} = \frac{3-0}{6-0} = \frac{3}{6} = \frac{1}{2}$ → slope of curve at $x=0$

However, "rate of change of the slope" = second derivative

$\frac{d^2y}{dx^2} = \frac{(2y-x)(\frac{dy}{dx}-2) - (y-2x)(2\frac{dy}{dx}-1)}{(2y-x)^2} = \frac{(2\cdot 3 - 0)(\frac{1}{2}-2) - (3-2\cdot 0)(2\cdot\frac{1}{2}-1)}{(2\cdot 3-0)^2}$

$\frac{d^2y}{dx^2} = \frac{6(-3/2) - 3\cancel{(0)}}{6^2} = \frac{-9}{36} = \boxed{\frac{-1}{4}}$

8. Given: $(g(x))^2 + (h(x))^2 = 1$, $g'(x) = (h(x))^2$, $h(x) > 0$, $g(0) = 0$

a) Justify that $h'(x) = -g(x)h(x)$

$(g(x))^2 + (h(x))^2 = 1$

$2g(x)\cdot g'(x) + 2h(x)\cdot h'(x) = 0 \rightarrow h'(x) = \frac{-\cancel{2}g(x)g'(x)}{\cancel{2}h(x)} = \frac{-g(x)\cdot g'(x)}{h(x)}$

$h'(x) = \frac{-g(x)\cdot(h(x))^{\cancel{2}}}{\cancel{h(x)}} \rightarrow \boxed{h'(x) = -g(x)\cdot h(x)}$

b) Justify that $h(x)$ has a rel. max at $x=0$

$h'(x) = -g(x)\cdot h(x) = 0$ at max $\rightarrow g(x)\cdot h(x) = 0 \rightarrow g(x) = 0$ at $x=0$ (given) or $h(x) = 0$ never bc $h(x) > 0$ for all x

$x=0$ is a critical point

Show that $x=0$ is a max (not min)

$h''(x) = -g(x)\cdot h'(x) + -h(x)\cdot g'(x)$

$h''(0) = -\underbrace{g(0)}_{0}\cdot h'(0) - h(0)\cdot g'(0) = -h(0)\cdot(h(0))^2 = -(h(0))^3 = -(+)^3 = -$

h is concave down at $x=0$, so the CP is a $\boxed{\text{max}}$.

8. c) Justify that g has POI at $x=0$

$g'(x) = (h(x))^2$

$g''(x) = 2h(x)h'(x) = 2h(x)(-g(x)\cdot h(x)) = -2g(x)\cdot(h(x))^2 = 0$

$g(x)\cdot(h(x))^2 = 0 \rightarrow g(x)=0$ at $x=0$ or $(h(x))^2 = 0$ never bc $h(x)>0$ for all x

$g''(x)$ must change signs at $x=0$ in order to be a POI

$g''(x) = -2g(x)\cdot(h(x))^2$

We know $g'(x) = (h(x))^2$, so $g(x) = \frac{1}{3}(h(x))^3 + C$, which is an odd function

$g''(x) = \underbrace{-2}_{\text{Always negative}}\cdot g(x)\cdot\underbrace{(h(x))^2}$ ← squared, so always positive

↑ Odd function (shown above), so changes signs at $x=0$

Therefore $g''(x)$ changes signs at $x=0$, and $x=0$ is a POI.

9. $f(x) = xe^{-2x} = \frac{x}{e^{2x}}$ on $[0,10]$

a) $f'(x) = \frac{e^{2x}\cdot 1 - x\cdot e^{2x}\cdot 2}{(e^{2x})^2} = \frac{\cancel{e^{2x}}(1-2x)}{\cancel{e^{2x}}\cdot e^{2x}} = \frac{1-2x}{e^{2x}}$

$1-2x=0$ when $2x=1 \rightarrow x=1/2$

e^{2x} ← never $=0$, so always exists

$f'(x)$: | + + + 0 − − − | $\frac{1-2x}{e^{2x}}$

$x=0$ End, $x=\frac{1}{2}$, $x=10$ End

$f'(0) = \frac{1-0}{e^0} = \frac{1}{1} = +$

$f'(1) = \frac{1-2}{e^2} = \frac{-1}{e^2} = \frac{-}{+} = -$

Inc: $(-\infty, 1/2)$ and Dec: $(1/2, \infty)$ if unlimited domain

With $[0,10] \rightarrow$ Inc: $\boxed{(0, 1/2)}$ and Dec: $\boxed{(1/2, 10)}$

b) Rel. max at $x=1/2$ bc sign of f' changes from + to −

Rel. min at $x=0$ bc f inc. after left endpoint

Rel. min at $x=10$ bc f dec. before right endpoint

Check y-values to determine absolute min/max

$f(0) = \frac{0}{e^0} = \frac{0}{1} = 0 \rightarrow$ $\boxed{(0,0) \text{ Abs. Min}}$

$f(1/2) = \frac{1/2}{e^1} = \frac{1}{2e} \rightarrow$ $\boxed{\left(\frac{1}{2}, \frac{1}{2e}\right) \text{ Abs. Max}}$

$f(10) = \frac{10}{e^{20}} \rightarrow (10, 10/e^{20})$ Rel. min, but not abs. min

10. $f(x) = (x^2+1)e^{-x} = \dfrac{x^2+1}{e^x}$ on $[-4,4]$

a) $f'(x) = \dfrac{e^x \cdot 2x - (x^2+1)e^x}{(e^x)^2} = \dfrac{\cancel{e^x}(2x-x^2-1)}{\cancel{e^x}\cdot e^x} = \dfrac{-x^2+2x-1}{e^x}$ ← never = 0

$-x^2+2x-1=0 \rightarrow x^2-2x+1=0 \rightarrow (x-1)(x-1)=0 \rightarrow (x-1)^2=0,\ x=1$

$f'(x)$: $- - - - - \overset{0}{|} - - -$

$x=-4$ End; $x=1$ Neither; $x=4$ End

$\dfrac{-x^2+2x-1}{e^x}$ $\quad f'(0) = \dfrac{0+0-1}{e^0} = \dfrac{-1}{1} = -$

$f'(2) = \dfrac{-4+4-1}{e^2} = \dfrac{-1}{e^2} = \dfrac{-}{+} = -$

Abs. Max at $\boxed{x=-4}$ bc f dec. after left endpoint.
Abs. Min at $x=4$ bc f dec. before right endpoint.
$f(-4) = \dfrac{16+1}{e^{-4}} = 17e^4 \rightarrow (-4, 17e^4)$

b) $f''(x) = \dfrac{e^x(-2x+2)-(-x^2+2x-1)e^x}{(e^x)^2} = \dfrac{\cancel{e^x}(-2x+2+x^2-2x+1)}{\cancel{e^x}\cdot e^x} = \dfrac{x^2-4x+3}{e^x}$ ← never = 0

$x^2-4x+3=0 \rightarrow (x-3)(x-1)=0 \rightarrow x=1,3$

$f''(x)$: $+ + + + \overset{0}{|} - - - \overset{0}{|} + +$ $\quad \dfrac{(x-3)(x-1)}{e^x}$

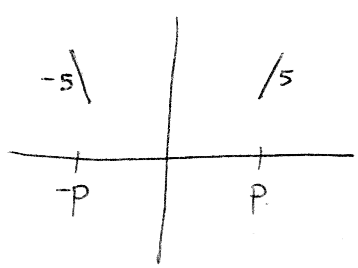
$x=-4$ End; $x=1$; $x=3$; $x=4$ End

$f''(0) = \dfrac{(-)(-)}{+} = \dfrac{+}{+} = +$ / $f''(2) = \dfrac{(-)(+)}{(+)} = -$ / $f''(3.5) = \dfrac{(+)(+)}{+} = +$

POI at $\boxed{x=1, x=3}$ bc sign of f'' changes there.

11. $f(-x) = f(x) \rightarrow$ even, $f(p)=1$, $f'(p)=5$ for $p>0$

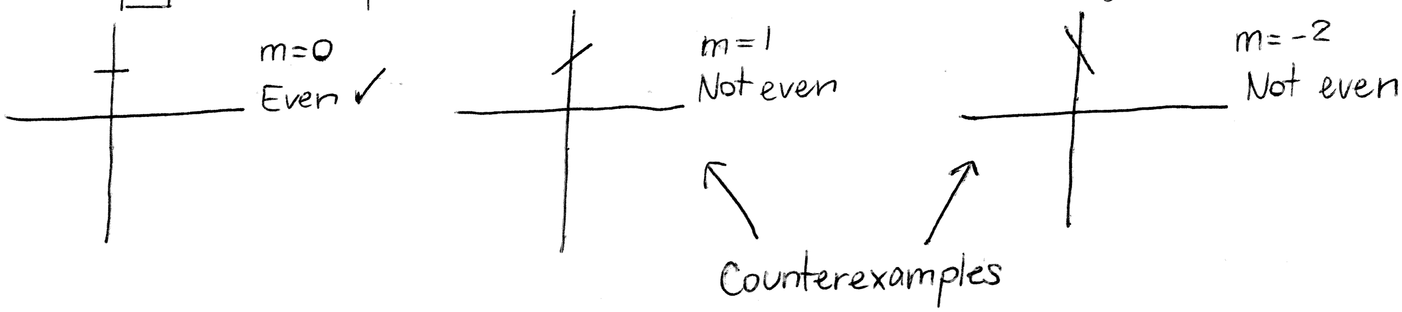
a) An even function is symmetric to the y-axis



$f'(-p)$ = slope at opposite of p
$f'(-p) = \boxed{-5}$ to maintain even symmetry

11. b) $f'(0) =$ slope at $x = 0$

$f'(0) = \boxed{0}$ bc a slope of 0 at $x=0$ maintains even symmetry



c) ℓ_1 tangent at $(-p, 1)$ and ℓ_2 tangent at $(p, 1)$

$\ell_1 \downarrow m = -5$ $\qquad \ell_2 \downarrow m = 5$

$\ell_1 : y - 1 = -5(x+p) \rightarrow y - 1 = -5x - 5p \rightarrow y = -5x - 5p + 1$

$\ell_2 : y - 1 = 5(x-p) \rightarrow y - 1 = 5x - 5p \rightarrow y = 5x - 5p + 1$

Intersection means $\ell_1 = \ell_2$: $-5x - 5p + 1 = 5x - 5p + 1$

$-5x = 5x$

$-10x = 0 \rightarrow x = 0$

$\ell_1 : y(0) = -5(0) - 5p + 1 = -5p + 1$ ✓

$\ell_2 : y(0) = 5(0) - 5p + 1 = -5p + 1$ ✓

$\boxed{(0, -5p+1)}$

12. $f(x) = 2\ln(x^2+3) - x$ on $[-3, 5]$

a) $f'(x) = \dfrac{2(2x)}{x^2+3} - 1 = \dfrac{4x}{x^2+3} - 1 = 0 \rightarrow \dfrac{4x}{x^2+3} = 1 \rightarrow 4x = x^2 + 3$

$x^2 - 4x + 3 = 0 \rightarrow (x-3)(x-1) = 0 \rightarrow x = 1, x = 3$

$f'(x)$ $\quad$ [− − − − − 0 + + 0 − −] $\qquad \dfrac{4x}{x^2+3} - 1$

$x=-3$ End $\qquad x=1 \qquad x=3 \qquad x=5$ End

$f'(0) = \dfrac{0}{3} - 1 = 0 - 1 = -$

$f'(2) = \dfrac{8}{7} - 1 = \dfrac{1}{7} = +$

$f'(4) = \dfrac{16}{19} - 1 = \dfrac{-3}{19} = -$

Rel. max at $x=-3$	bc f dec. after left endpoint
Rel. min at $x=1$	bc sign of f' changes from − to +
Rel. max at $x=3$	bc sign of f' changes from + to −
Rel. min at $x=5$	bc f dec. before right endpoint

12. b) $f'(x) = \frac{4x}{x^2+3} - 1$

$$f''(x) = \frac{(x^2+3)4 - (4x)2x}{(x^2+3)^2} = \frac{4x^2+12-8x^2}{(x^2+3)^2} = \frac{12-4x^2}{(x^2+3)^2}$$

$12 - 4x^2 = 0 \rightarrow 4x^2 = 12 \rightarrow x^2 = 3 \rightarrow x = \pm\sqrt{3}$

$f''(x)$ — — — 0 + + + 0 — — $\quad \frac{4(3-x^2)}{(x^2+3)^2}$

$x=-3$ End $\quad x=-\sqrt{3} \quad x=\sqrt{3} \quad x=5$ End

$$f''(-2) = \frac{4(3-4)}{+} = \frac{-}{+} = -$$

$$f''(0) = \frac{4(3-0)}{+} = \frac{+}{+} = +$$

$$f''(2) = \frac{4(3-4)}{+} = \frac{-}{+} = -$$

POI at $\boxed{x = \pm\sqrt{3}}$

c) Rel. max at $x=-3$, $x=3$ from part a

Check y-values to determine abs. max

$f(x) = 2\ln(x^2+3) - x$

$f(-3) = 2\ln(9+3) + 3 = 2\ln(12) + 3 \leftarrow$ larger, so Abs. Max $= \boxed{2\ln(12)+3}$

$f(3) = 2\ln(9+3) - 3 = 2\ln(12) - 3 \leftarrow$ smaller