Optimization - Day 2: #13-24

13. $f(x) = \sin^2 x - \sin x$ on $[0, 3\pi/2]$

a) x-int when $y = 0$

$\sin^2 x - \sin x = 0 \to \sin x(\sin x - 1) = 0 \to \sin x = 0$ or $\sin x - 1 = 0$

$\boxed{x = 0, \pi}$ $\quad$ $\sin x = 1$ $\quad$ $\boxed{x = \pi/2}$

b) $f'(x) = 2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0 \to \cos x = 0$ or $2\sin x - 1 = 0$

$x = \pi/2, 3\pi/2$ $\quad$ $\sin x = 1/2$

$x = \pi/6, 5\pi/6$

$f'(x)$: $\quad$ $x=0$ $[\ - - -\ ]$ $x=\frac{\pi}{6}$ (0) $[\ + +\ ]$ $x=\frac{\pi}{2}$ (0) $[\ - - -\ ]$ $x=\frac{5\pi}{6}$ (0) $[\ + + +\ ]$ $x=\frac{3\pi}{2}$ (0) $\quad$ $\cos x(2\sin x - 1)$

$f'(\pi/12) = \cos\frac{\pi}{12}(2\sin\frac{\pi}{12} - 1) \approx -0.466 = -$

$f'(\pi/4) = \frac{\sqrt{2}}{2}(\cancel{2}\frac{\sqrt{2}}{\cancel{2}} - 1) = \frac{\sqrt{2}}{2}(\sqrt{2} - 1) = (+)(+) = +$

$f'(\pi) = -1(2 \cdot 0 - 1) = -1(-1) = 1 = +$

$f'(3\pi/4) = -\frac{\sqrt{2}}{2}(\cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} - 1) = -\frac{\sqrt{2}}{2}(\sqrt{2} - 1) = (-)(+) = -$

Inc: $\boxed{(\pi/6, \pi/2) \cup (5\pi/6, 3\pi/2)}$

c) Rel. max at $x = 0$ bc f dec. after left endpoint

Rel. max at $x = \pi/2$ bc sign of f' changes + to -

Rel. max at $x = 3\pi/2$ bc f inc. before right endpoint

$f(0) = (\sin 0)^2 - \sin 0 = 0^2 - 0 = 0 \to (0,0)$

$f(\pi/2) = (\sin\pi/2)^2 - \sin\pi/2 = 1^2 - 1 = 0 \to (\pi/2, 0)$

$f(3\pi/2) = (\sin 3\pi/2)^2 - \sin 3\pi/2 = (-1)^2 - -1 = 1 + 1 = 2 \to (3\pi/2, 2) \to \boxed{\text{Abs. Max} = 2}$

Rel. min at $x = \pi/6$ bc sign of f' changes - to +

Rel. min at $x = 5\pi/6$ bc sign of f' changes - to +

$f(\pi/6) = (\sin\pi/6)^2 - \sin\pi/6 = (1/2)^2 - 1/2 = 1/4 - 1/2 = -1/4 \to (\pi/6, -1/4)$

$f(5\pi/6) = (\sin 5\pi/6)^2 - \sin 5\pi/6 = (1/2)^2 - 1/2 = 1/4 - 1/2 = -1/4 \to (5\pi/6, -1/4)$

$\boxed{\text{Abs. Min} = -1/4}$

14. $f(x) = \ln\left|\frac{x}{1+x^2}\right|$

a) $1+x^2 \neq 0$, so not a problem

Domain of ln is only positive values, so $\frac{x}{1+x^2} \neq 0$, so $x \neq 0$

Domain: $\boxed{(-\infty, 0) \cup (0, \infty)}$

b) Example: $x=1$: $\ln\left|\frac{1}{1+1^2}\right| = \ln\left|\frac{1}{2}\right| = \ln(1/2)$

$x=-1$: $\ln\left|\frac{-1}{1+(-1)^2}\right| = \ln\left|\frac{-1}{2}\right| = \ln(1/2)$

$f(-x) = f(x)$, so $\boxed{\text{even}}$

c) For $x>0$, $f(x) = \ln\left(\frac{x}{1+x^2}\right)$

$$f'(x) = \frac{1}{\left(\frac{x}{1+x^2}\right)} \cdot \frac{(1+x^2)(1)-(x)(2x)}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^{\cancel{2}}} \cdot \frac{\cancel{1+x^2}}{x} = \frac{1-x^2}{x(1+x^2)}$$

$f'(x) = 0$ when $1-x^2 = 0 \to x^2 = 1 \to x = \pm 1$

$f'(x)$ DNE when $x(1+x^2) = 0 \to x = 0$ or $1+x^2 = 0$ (never)

$f'(x)$: $+ + + \; 0 \; - - - \; \text{DNE} \; + + \; 0 \; - - -$ at $x=-1$, $x=0$, $x=1$; $\frac{1-x^2}{x(1+x^2)^2}$ for $x>0$

$f'(1/2) = \frac{1-1/4}{\frac{1}{2}(1+1/4)^2} = \frac{+}{(+)(+)} = +$

Fill in sign line for $x<0$ by even symmetry.

$f'(2) = \frac{1-4}{2(1+4)^2} = \frac{-}{(+)(+)} = -$

$\boxed{\text{Rel. max at } x=1, x=-1}$ bc sign of f' changes from $+$ to $-$

$\boxed{\text{No rel. min anywhere}}$ bc sign of f' changes from $-$ to $+$ at $x=0$, but we cannot use $x=0$ bc it isn't in the domain of f.

d) $f(1) = \ln\left|\frac{1}{1+1^2}\right| = \ln\left|\frac{1}{2}\right| = \ln(1/2)$

$\lim_{x \to 0} f(x)$ approaches $-\infty$

Range: $\boxed{(-\infty, \ln 1/2]}$

15. $f(x) = x^3 - 5x^2 + 3x + k$

a) $f'(x) = 3x^2 - 10x + 3 = 0$

$3x^2 - 9x - x + 3 = 0 \to 3x(x-3) - 1(x-3) = 0 \to (x-3)(3x-1) = 0 \to x = 3, 1/3$

$f'(x)$: $+ + 0 - - - 0 + + +$ $\quad 3x^2 - 10x + 3$

$x = \frac{1}{3}$, $x = 3$

$f'(0) = 0 - 0 + 3 = 3 = +$

$f'(1) = 3 - 10 + 3 = -4 = -$

$f'(4) = 48 - 40 + 3 = 11 = +$

Inc : $\boxed{(-\infty, 1/3) \cup (3, \infty)}$

b) $f''(x) = 6x - 10 = 0$ at $x = 10/6 = 5/3$

$f''(x)$: $- - - 0 + + +$ $\quad 6x - 10$

$x = \frac{5}{3}$

CC down : $\boxed{(-\infty, 5/3)}$

c) From sign line in part a, rel. min at $x = 3$ (sign of f' $-$ to $+$)

$f(3) = 11$, solve for k

$3^3 - 5 \cdot 3^2 + 3 \cdot 3 + k = 11 \to 27 - 45 + 9 + k = 11 \to -9 + k = 11 \to \boxed{k = 20}$

16. $f(x) = 2x \cdot e^{2x}$

a) $\lim_{x \to \infty} 2x \cdot e^{2x} = 2\infty \cdot e^{2\infty} = \boxed{\infty}$

$\lim_{x \to -\infty} 2x \cdot e^{2x} = 2(-\infty) \cdot e^{2(-\infty)} = -2\infty \cdot e^{-2\infty} = \frac{-2\infty}{e^{2\infty}} = \boxed{0}$

big

bigger

b) No endpoints

$f'(x) = 2x \cdot e^{2x} \cdot 2 + e^{2x} \cdot 2 = 4xe^{2x} + 2e^{2x} = 2e^{2x}(2x+1) = 0$

$2e^{2x} = 0 \to$ never

$2x + 1 = 0 \to x = -1/2$

$f'(x)$: $- - - 0 + + +$ $\quad 2e^{2x}(2x+1)$

$x = -\frac{1}{2}$

$f'(0) = 2e^0(0+1) = (+)(+) = +$

$f'(-1) = 2e^{-2}(-2+1) = \frac{-2}{e^2} = \frac{-}{+} = -$

16. b) continued

$f(-1/2) = 2 \cdot \frac{-1}{2} \cdot e^{2(-1/2)} = -e^{-1} = \frac{-1}{e}$ (less than 0 from part a)

Sign of f' changes from $-$ to $+$ at $x = -1/2 \rightarrow$ $\boxed{\text{Abs. Min} = -1/e}$

c) Max is ∞ from part a

Range: $\boxed{[-1/e, \infty)}$

d) $y = bx \cdot e^{bx}$

$y' = bx \cdot e^{bx} \cdot b + e^{bx} \cdot b = b^2 x e^{bx} + be^{bx} = be^{bx}(bx+1) = 0$

$be^{bx} \neq 0$ or $bx + 1 = 0 \rightarrow x = -1/b$

$y(\frac{-1}{b}) = \cancel{b}(\frac{-1}{\cancel{b}}) \cdot e^{\cancel{b}(-1/\cancel{b})} = -e^{-1} = \frac{-1}{e}$

In general, abs. min at $\boxed{(-1/b, -1/e)}$

17. a) Rel. min when f' changes $-$ to $+$: $\boxed{x = -1}$

Rel. min when f incr. after left endpoint: $\boxed{x = -7}$

b) Rel. max when f' changes $+$ to $-$: $\boxed{x = -5}$

Rel. max when f incr. before right endpoint: $\boxed{x = 7}$

c) $f''(x) < 0$ (concave down) when f' is decreasing: $(-7,-3) \cup (2,5)$

Except, exclude $x=3$ bc slope of f' DNE at $x=3$, so $f''(3)$ DNE

$\boxed{(-7,-3) \cup (2,3) \cup (3,5)}$

d) We know there are relative maximums at both $x=-5$ and $x=7$, but without any y-values of f, we $\boxed{\text{cannot determine}}$ the absolute maximum.

18. a) $x^2 + 4y^2 = 7 + 3xy$

$2x + 8y \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$

$8y \frac{dy}{dx} - 3x \frac{dy}{dx} = 3y - 2x$

$\frac{dy}{dx} = \boxed{\frac{3y - 2x}{8y - 3x}}$

18. b) $x=3$

$3^2+4y^2=7+3(3)y \rightarrow 9+4y^2=7+9y \rightarrow 4y^2-9y+2=0$

$4y^2-8y-y+2=0 \rightarrow 4y(y-2)-1(y-2)=0 \rightarrow (y-2)(4y-1)=0 \rightarrow y=2, 1/4$

$(3,2)$ or $(3,1/4) \rightarrow$ check in $\frac{dy}{dx}$ for 0 slope

$$\frac{dy}{dx}=\frac{3y-2x}{8y-3x} \text{ at } (3,1/4) = \frac{3/4-6}{2-9}=\frac{-5.25}{-7}\neq 0$$

$$\frac{dy}{dx}=\frac{3y-2x}{8y-3x} \text{ at } \boxed{(3,2)} = \frac{6-6}{16-9}=\frac{0}{7}=0 \ \checkmark$$

c) $P(3,2)$

$$\frac{d^2y}{dx^2}=\frac{(8y-3x)(3\frac{dy}{dx}-2)-(3y-2x)(8\frac{dy}{dx}-3)}{(8y-3x)^2}$$

$\frac{d^2y}{dx^2}$ evaluated at $x=3$, $y=2$, $\frac{dy}{dx}=0$

$$\frac{d^2y}{dx^2}=\frac{(16-9)(0-2)-(6-6)(0-3)}{(16-9)^2}=\frac{7(-2)-\cancel{0(-3)}}{7^2}=\frac{-14}{49}=\frac{-2}{7}= -$$

At $(3,2)$, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$, so the graph is concave down and therefore, $(3,2)$ is a $\boxed{\text{local max}}$.

19. Given: $f(0)=0$, $f'(0)=0$, $f(4)=1$, $f'(4)=1$, Inc on $(0,4)$

a) $f(x)=ax^2$

$f(4)=a\cdot 4^2=1 \rightarrow a=1/16$

$f'(x)=2ax$

$f'(4)=2a(4)=1 \rightarrow a=1/8$

$\boxed{\frac{1}{16}\neq\frac{1}{8}}$, so not a consistent value of a that fits $f(4)=1$ and $f'(4)=1$.

b) $g(x)=cx^3-\frac{1}{16}x^2$

$g(4)=c\cdot 4^3-\frac{1}{16}\cdot 4^2=1 \rightarrow 64c-1=1 \rightarrow 64c=2 \rightarrow c=1/32$

$g'(x)=3cx^2-\frac{1}{8}x$

$g'(4)=3c\cdot 4^2-\frac{1}{8}\cdot 4=1 \rightarrow 48c-\frac{1}{2}=1 \rightarrow 48c=1.5 \rightarrow \boxed{c=1/32}$

19. c) $g(x) = \frac{1}{32}x^3 - \frac{1}{16}x^2$

$g'(x) = \frac{3}{32}x^2 - \frac{1}{8}x = 0 \rightarrow \frac{3}{32}x^2 - \frac{4}{32}x = 0 \rightarrow \frac{1}{32}x(3x-4) = 0 \rightarrow x = 0, 4/3$

$g'(x)$: $x=0$ | $-\ \ -$ | $x=\frac{4}{3}$ (0) | $+\ \ +\ \ +$ | $x=4$

$g'(1) = \frac{3}{32} - \frac{4}{32} = \frac{-1}{32} = -$

$g'(2) = \frac{12}{32} - \frac{8}{32} = \frac{4}{32} = +$

Using this equation, $g(x)$ is decreasing on $(0, 4/3)$, so not increasing for all $(0,4)$ as stated in iii.

d) $h(x) = \frac{x^n}{k} = \frac{1}{k} \cdot x^n$

$h(4) = \frac{1}{k} \cdot 4^n = 1 \rightarrow 4^n = k$

$h(x) = \frac{x^n}{k} = \frac{x^n}{4^n} = \left(\frac{x}{4}\right)^n$

$h'(x) = n\left(\frac{x}{4}\right)^{n-1} \cdot \frac{1}{4}$

$h'(4) = \frac{n}{4} \cdot \left(\frac{4}{4}\right)^{n-1} = 1 \rightarrow \frac{n}{4} \cdot \cancel{1^{n-1}} = 1 \rightarrow \frac{n}{4} = 1 \rightarrow \boxed{n=4}$

$h(x) = \left(\frac{x}{4}\right)^n = \left(\frac{x}{4}\right)^4 = \frac{x^4}{4^4} \leftarrow k$ so $k = 4^4 \rightarrow \boxed{k = 256}$

$h(x) = \frac{x^4}{256}$

Check i: $f(0) = 0$ and $f'(0) = 0$

$h(0) = \frac{0^4}{256} = 0$ ✓

$h'(x) = \frac{4}{256}x^3 = \frac{1}{64}x^3$, so $h'(0) = \frac{1}{64} \cdot 0^3 = 0$ ✓

Check iii: Inc. on $(0,4)$

$h'(x) = \frac{1}{64}x^3 = 0$ when $x = 0$

$h'(x)$: $x=0$ | $+\ \ +\ \ +\ \ +\ \ +\ \ +$ | $x=4$ ✓

$h'(1) = \frac{1}{64} \cdot 1^3 = \frac{1}{64} = +$

20. $f(x) = k\sqrt{x} - \ln x$ for $x > 0$

$f(x) = kx^{1/2} - \ln x$

a) $f'(x) = \frac{1}{2}kx^{-1/2} - \frac{1}{x} = \boxed{\frac{k}{2\sqrt{x}} - \frac{1}{x}}$

$f''(x) = -\frac{1}{4}kx^{-3/2} + 1x^{-2} = \boxed{\frac{-k}{4x^{3/2}} + \frac{1}{x^2}}$

b) $\frac{k}{2\sqrt{x}} - \frac{1}{x} = 0$ at $x=1 \rightarrow \frac{k}{2\sqrt{1}} - \frac{1}{1} = 0 \rightarrow \frac{k}{2} - 1 = 0 \rightarrow \frac{k}{2} = 1 \rightarrow \boxed{k=2}$

$f''(x) = \frac{-2}{4x^{3/2}} + \frac{1}{x^2} = \frac{-1}{2x^{3/2}} + \frac{1}{x^2}$

$f''(1) = \frac{-1}{2\cdot 1} + \frac{1}{1} = \frac{-1}{2} + 1 = \frac{1}{2} = + \rightarrow$ concave up $\rightarrow$ $\boxed{\text{rel. min}}$

c) $k = ?$

$f''(0) = 0 \rightarrow \frac{-k}{4x^{3/2}} + \frac{1}{x^2} = 0 \rightarrow \frac{1}{x^2} = \frac{k}{4x^{3/2}} \rightarrow 4x^{3/2} = kx^2 \rightarrow k = \frac{4x^{3/2}}{x^2} = 4x^{-1/2} = \frac{4}{\sqrt{x}}$

Point on y-axis, so $f(x) = 0 \rightarrow k\sqrt{x} - \ln x = 0 \rightarrow k\sqrt{x} = \ln x \rightarrow k = \frac{\ln x}{\sqrt{x}}$

Set two results above equal: $\frac{4}{\sqrt{x}} = \frac{\ln x}{\sqrt{x}} \rightarrow \cancel{\ln} x = e^4 \rightarrow x = e^4$

$k = \frac{4}{\sqrt{x}} = \frac{4}{\sqrt{e^4}} = \frac{4}{(e^4)^{1/2}} = \boxed{\frac{4}{e^2}}$

21. $f(2) = 5$, $f(5) = 2$, $g = f(f(x))$

a) Avg. ROC $= \frac{f(5) - f(2)}{5-2} = \frac{2-5}{5-2} = \frac{-3}{3} = -1$

f is differentiable on $(2,5)$ and continuous on $[2,5]$, so Mean Value Theorem says that for some c on $(2,5)$ the average ROC = the instantaneous slope, so $f'(c) = -1$.

21. b) $g(x) = f(f(x))$

$g'(x) = f'(f(x)) \cdot f'(x)$

$\left.\begin{array}{l} g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2) \\ g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5) \end{array}\right\}$ equivalent, so $g'(2) = g'(5)$ ✓

$$\text{Avg. ROC} = \frac{g'(5) - g'(2)}{5-2} = \frac{f'(2) \cdot f'(5) - f'(5) \cdot f'(2)}{5-2} = \frac{0}{3} = 0$$

g is differentiable on $(2,5)$ and continuous on $[2,5]$, so Mean Value Theorem says that for some k on $(2,5)$, the average ROC = the instantaneous slope, so $g''(k) = 0$.

c) $g'(x) = f'(f(x)) \cdot f'(x)$

$$g''(x) = f'(f(x)) \cdot \underbrace{f''(x)}_{0} + f'(x) \cdot \underbrace{f''(f(x))}_{0} \cdot f'(x) = 0 + 0 = 0$$

$g''(x) = 0$ for all x, so the sign of g'' never changes, so $g''(x)$ has no point of inflection.

d) $h(x) = f(x) - x$

$h(2) = f(2) - 2 = 5 - 2 = 3 \rightarrow (2,3)$

$h(5) = f(5) - 5 = 2 - 5 = -3 \rightarrow (5,-3)$

h is continuous on $[2,5]$, so Intermediate Value Theorem says that for some r on $(2,5)$ the y-value must pass through 0 between 3 and -3.

22. $f(0) = 2$, $f'(0) = -3$, $f''(0) = 0$

$$g'(x) = e^{-2x}(3f(x) + 2f'(x)) = \frac{3f(x) + 2f'(x)}{e^{2x}}$$

a) $(0,2)$, $m = -3 \rightarrow y = -3x + 2$

b) We know $f''(0) = 0$, but do not know if the sign of f'' actually changes.

c) $g(0) = 4$

$$g'(0) = \frac{3f(0) + 2f'(0)}{e^{2 \cdot 0}} = \frac{3(2) + 2(-3)}{e^0} = \frac{6-6}{1} = \frac{0}{1} = 0$$

$y - 4 = 0(x - 0) \rightarrow y - 4 = 0 \rightarrow \boxed{y = 4}$

22. d) $g'(x) = \dfrac{3f(x)+2f'(x)}{e^{2x}}$

$$g''(x) = \frac{e^{2x}(3f'(x)+2f''(x)) - (3f(x)+2f'(x))e^{2x}\cdot 2}{(e^{2x})^2}$$

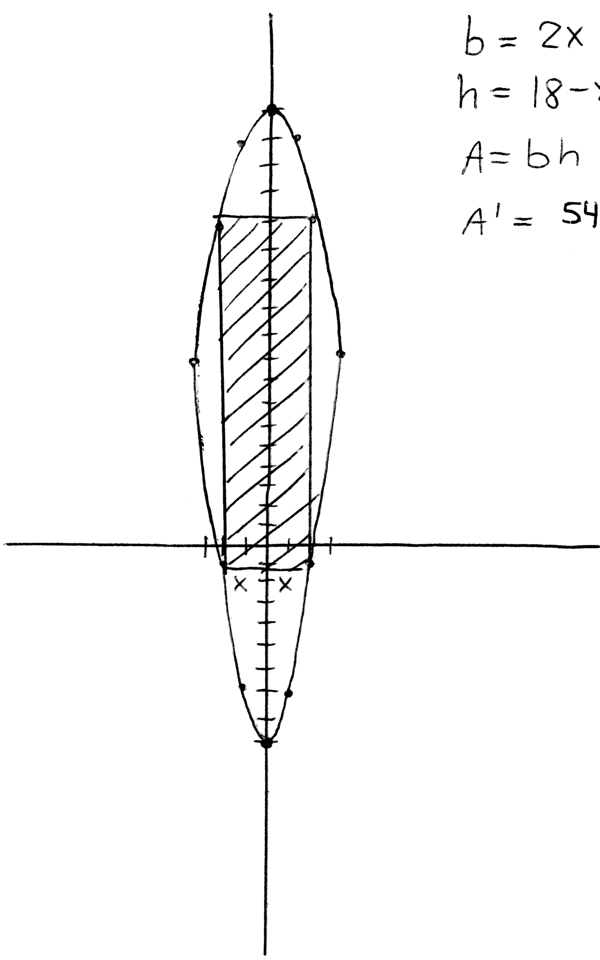
$$g''(x) = \frac{\cancel{e^{2x}}(3f'(x)+2f''(x)-6f(x)-4f'(x))}{\cancel{e^{2x}}\cdot e^{2x}} = \boxed{\frac{-6f(x)-f'(x)+2f''(x)}{e^{2x}}}$$

$$g''(0) = \frac{-6f(0)-f'(0)+2f''(0)}{e^0} = \frac{-6(2)-(-3)+2\cancel{(0)}}{1} = -12+3 = -9 = -$$

At $x=0$, $g'(0)=0$, so there is a critical point. $g''(0)<0$, so g is concave down at $x=0$. Therefore, the critical point is a local maximum.

23. $18-x^2 = 2x^2-9$

$27 = 3x^2 \rightarrow x^2 = 9 \rightarrow x = \pm 3$



$b = 2x$

$h = 18-x^2-2x^2+9 = -3x^2+27$

$A = bh = 2x(27-3x^2) = 54x-6x^3$

$A' = 54-18x^2 = 0$

$18(3-x^2) = 0$

$3-x^2 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$b = 2(\sqrt{3}) = 2\sqrt{3}$

$h = 27-3\cdot\cancel{\sqrt{3}^2} = 27-9 = 18$

$A = 2\sqrt{3}\cdot 18 = \boxed{36\sqrt{3}\text{ units}^2}$

24. $x(t) = (t-2)^3(t-6)$

a) $v(t) = (t-2)^3 \cdot 1 + (t-6) \cdot 3(t-2)^2 = (t-2)^2[t-2+3(t-6)]$

$v(t) = (t-2)^2(t-2+3t-18) = (t-2)^2(4t-20) = 0$

$(t-2)^2 = 0$ or $4t-20=0$

$t-2=0$ $\quad$ $4t=20$

$t=2$ $\quad$ $t=5$

$v(t)$: $- - - -$ | $t=2$ | $- - - -$ | $t=5$ (∪) | $+ + +$, starting at $t=0$

$v(1) = (+)(-) = -$

$v(3) = (+)(-) = -$

$v(6) = (+)(+) = +$

Moving right when $v(t) > 0$, so $\boxed{(5, \infty)}$

b) At rest when $v(t) = 0$, so $\boxed{t = 2, 5}$

c) Particle changes direction at $\boxed{t=5}$ bc sign of v changes $-$ to $+$

d) Farthest left = minimum position at $\boxed{t=5}$ bc sign of v changes from $-$ to $+$, so particle switches from moving left before $t=5$ to moving right after $t=5$.

$x(5) = (5-2)^3(5-6) = 3^3 \cdot -1 = -27 =$ $\boxed{\text{27 units left}}$