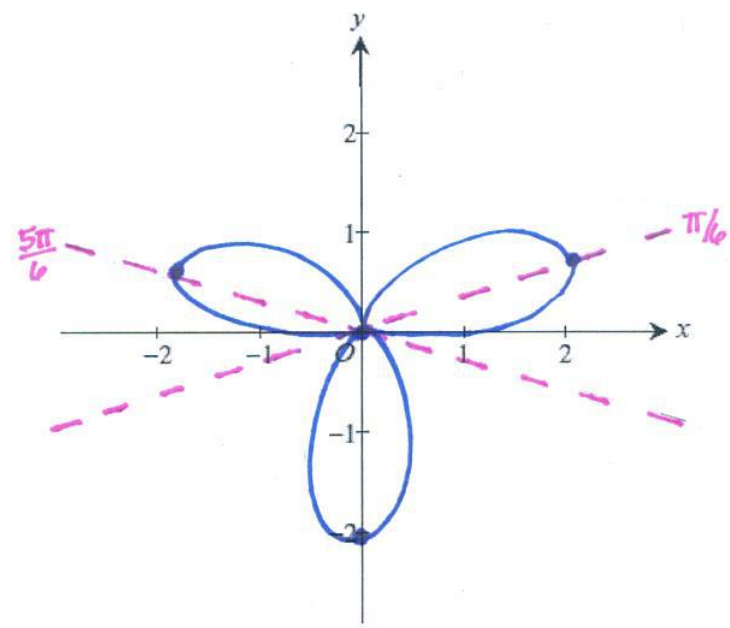
AP CALCULUS FREE RESPONSE QUESTIONS – POLAR FUNCTIONS

Please do not write on this packet. Show all work on separate paper.

1993 #4 – no calculator

Consider the polar curve $r = 2\sin(3\theta)$ for $0 \le \theta \le \pi$.

(a) In the xy-plane provided below, sketch the curve.



θ	$r = 2\sin 3\theta$
0	$2\sin 0 = 0$
$\pi/6$	$2\sin 3\pi/6 = 2$
$\pi/3$	$2\sin\pi = 0$
$\pi/2$	$2\sin 3\pi/2 = -2$
$5\pi/6$	$2\sin 5\pi/2 = 2$
π	$2\sin 3\pi = 0$

(b) Find the area of the region inside the curve.

(c) Find the slope of the curve at the point where $\theta = \frac{\pi}{4}$.

b) $A = \int_0^{\pi} \frac{1}{2}(2\sin 3\theta)^2\, d\theta = \int_0^{\pi} 2\sin^2 3\theta\, d\theta = 2\left[\frac{\theta}{2} - \frac{\sin 6\theta}{12}\right]\Big|_0^{\pi}$

use $\int$ table #58

$= \theta - \frac{\sin 6\theta}{6}\Big|_0^{\pi}$

$= (\pi - 0) - (0 - 0) \boxed{= \pi}$

$x = 2\sin 3\theta \cdot \cos\theta$

c) $y = 2\sin 3\theta \cdot \sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin 3\theta(\cos\theta) + \sin\theta(6\cos 3\theta)}{2\sin 3\theta(-\sin\theta) + \cos\theta(6\cos 3\theta)}$

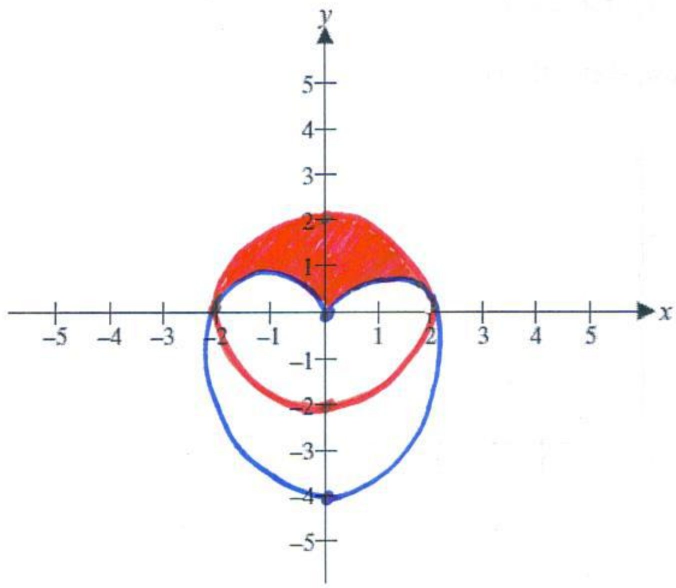
@ $\theta = \pi/4$ $= \frac{2(\sqrt{2}/2)(\sqrt{2}/2) + (\sqrt{2}/2)(6)(-\sqrt{2}/2)}{2(\sqrt{2}/2)(-\sqrt{2}/2) + (\sqrt{2}/2)(6)(-\sqrt{2}/2)} = \frac{1 + (-3)}{-1 + (-3)} = \frac{-2}{-4} = \boxed{\frac{1}{2}}$

1990 #4 – no calculator

Let R be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin\theta)$.

$r = 2 - 2\sin\theta$

(a) Sketch the two polar curves in the xy-plane provided below and shade the region R



θ	$r=2$	$r=2-2\sin\theta$
0	2	$2-2\sin 0=2$
$\pi/2$	2	$2-2\sin\pi/2=0$
π	2	$2-2\sin\pi=2$
$3\pi/2$	2	$2-2\sin 3\pi/2=4$
2π	2	$2-2\sin 2\pi=0$

(b) Find the area of R.

$$A = \int_0^{\pi} \frac{1}{2}\left[(2)^2 - (2-2\sin\theta)^2\right] d\theta = \cancel{2}\int_0^{\pi/2} \frac{1}{\cancel{2}}\left[\cancel{4} - \cancel{4} + 8\sin\theta - 4\sin^2\theta\right] d\theta$$

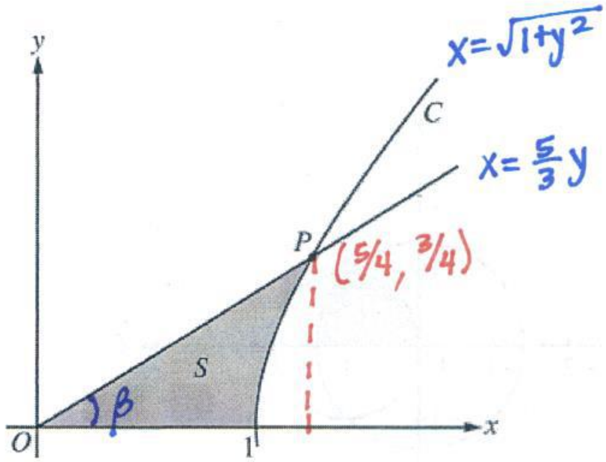
$$= \int_0^{\pi/2} \left(8\sin\theta - 4\sin^2\theta\right) d\theta$$ use $\int$ table #58

$$= -8\cos\theta - \left[\frac{4\theta}{2} - \frac{4\sin 2\theta}{2}\right]\Big|_0^{\pi/2}$$

$$= \left[-8\cos\frac{\pi}{2} - 2\left(\frac{\pi}{2}\right) + 2\sin\pi\right] - \left[-8\cos 0 - 2(0) + 2\sin 0\right]$$

$$\boxed{= -\pi + 8}$$

2003 #3 – calculator



3. The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

a) $\frac{5}{3}y = \sqrt{1+y^2}$

$\frac{25}{9}y^2 = 1+y^2$

$\frac{16}{9}y^2 = 1$

$y^2 = \frac{9}{16} \rightarrow y = 3/4$

$x = \frac{5}{3}\left(\frac{3}{4}\right) = \frac{5}{4}$

$\boxed{\therefore P(5/4, 3/4)}$

$\frac{dx}{dy} = \frac{1}{2}(1+y^2)^{-1/2}(2y)$

$= \frac{y}{(1+y^2)^{1/2}}$ @ P

$= \frac{3/4}{(1+9/16)^{1/2}} = \frac{3/4}{5/4}$

$\boxed{\frac{dx}{dy} = 3/5}$

b) $A = \int_0^{3/4}\left[\sqrt{1+y^2} - \frac{5}{3}y\right]dy \; \boxed{= 0.347}$

c) $x = r\cos\theta$

$y = r\sin\theta$

$x^2 - y^2 = 1$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r^2(\cos^2\theta - \sin^2\theta) = 1$

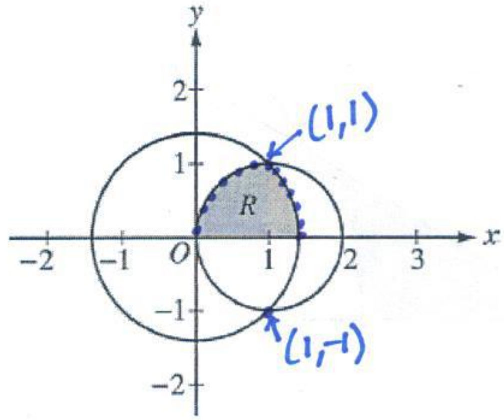
$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$ ✓

d) $\tan\beta = y/x = \frac{3/4}{5/4} = \frac{3}{5}$

$\beta = \tan^{-1}(3/5)$

$\boxed{A = \int_0^{\beta} \frac{1}{2}\left(\frac{1}{\cos^2\theta - \sin^2\theta}\right)d\theta}$

2003B #2 – calculator



2. The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1, 1)$ and $(1, -1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.

   (a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

   (b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

   (c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

a) $\text{Area} = \int_0^1 \sqrt{(1)^2 - (x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

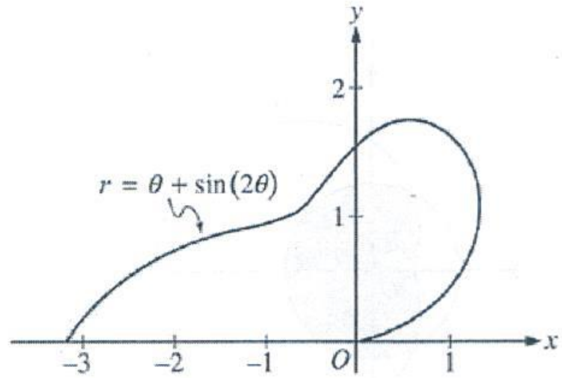
OR $\text{Area} = \frac{1}{4}(\pi \cdot 1^2) + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

b) $\text{Area} = \int_0^1 \left[\sqrt{2-y^2} - \left(1-\sqrt{1-y^2}\right)\right] dy$

c) $\text{Area} = \int_0^{\pi/4} \frac{1}{2}(\sqrt{2})^2\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

OR $\frac{1}{4}\left[\pi \cdot (\sqrt{2})^2\right] + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

2005 #2 – calculator



2. The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

a) $A = \int_0^{\pi} \frac{1}{2}(\theta + \sin 2\theta)^2 \, d\theta = \boxed{4.382}$

b) $x = -2$ $\quad \underbrace{-2}_{y_1} = \underbrace{(\theta + \sin 2\theta)\cos\theta}_{y_2}$ ← y_2 find intersection in calc.

$x = r\cos\theta$ $\quad \boxed{\theta = 2.786}$

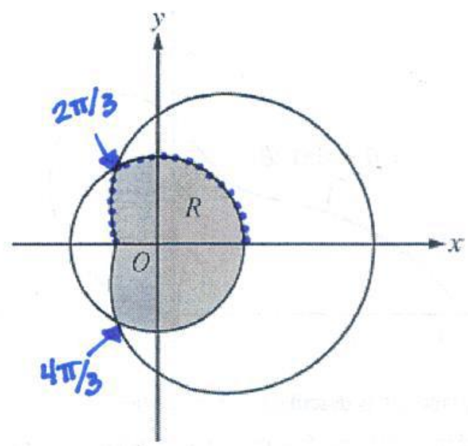
c) $\frac{dr}{d\theta} < 0$ for $\pi/3 < \theta < 2\pi/3$, then r is decreasing on this interval so the curve is getting closer to the origin.

d) $\frac{dr}{d\theta} = 0$ @ $\theta = \pi/3$ on the interval $[0, \pi/2]$

θ	r
0	0
$\pi/3$	1.913
$\pi/2$	1.571

∴ The greatest distance occurs @ $\theta = \pi/3$.

2007 #3 – calculator



3. The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

   (a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

   (b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

   (c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

a) $A = \int_0^{2\pi/3} \frac{1}{2}(2)^2\,d\theta + \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(3+2\cos\theta)^2\,d\theta = 10.370$

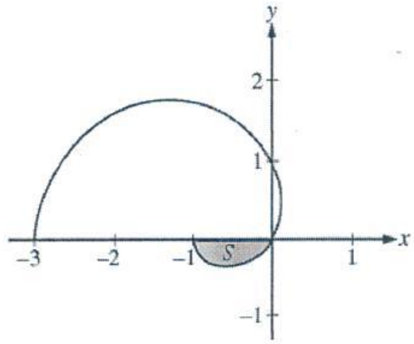
b) $\frac{dr}{dt} = \frac{dr}{d\theta} = -2\cos\theta$ @ $\theta = \pi/3 \rightarrow -2\cos(\pi/3) \approx -1.732$

$\therefore$ the particle is moving closer to the origin @ $\theta = \pi/3$.
($dr/dt < 0$ and $r > 0$)

c) $y = r\sin\theta$
$y = (3+2\cos\theta)\sin\theta$ $\quad \frac{dy}{d\theta}$ @ $\theta = \pi/3 = 0.5$

$\therefore$ the particle is moving away from the x-axis @ $\theta = \pi/3$
($dy/dt > 0$ and $y > 0$)

2009B #4 – no calculator



4. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

   (a) Write an integral expression for the area of S.

   (b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

   (c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

a) $r(0) = -1$ $\qquad A = \int_0^{\pi/3} \frac{1}{2}(1-2\cos\theta)^2\, d\theta$

$r(\pi/3) = 0$

b) $x = r\cos\theta = (1-2\cos\theta)\cos\theta$

$y = r\sin\theta = (1-2\cos\theta)\sin\theta$

$dx/d\theta = (1-2\cos\theta)(-\sin\theta) + \cos\theta(2\sin\theta)$

$\boxed{= 4\sin\theta\cos\theta - \sin\theta}$

$dy/d\theta = (1-2\cos\theta)(\cos\theta) + \sin\theta(2\sin\theta)$

$\boxed{= \cos\theta - 2\cos^2\theta + 2\sin^2\theta}$

c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\cos^2\theta + 2\sin^2\theta}{4\sin\theta\cos\theta - \sin\theta}$ $\quad @\ \theta = \pi/2 \rightarrow \frac{dy}{dx} = \frac{0 - 2(0)^2 + 2(1)^2}{4(0) - (1)}$

$\boxed{\frac{dy}{dx} = -2}$

$@\ \theta = \pi/2 \rightarrow \boxed{\begin{matrix} x = 0 \\ y = 1 \end{matrix}}$ $\quad \therefore\ y - 1 = -2(x - 0)$

$\boxed{y = -2x + 1}$