Section 10.1 : 1–25 odd

1. $\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

a) $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{*}\right) = \underset{n=1}{\frac{1}{1^2}} - \underset{n=2}{\frac{1}{2^2}} + \underset{n=3}{\frac{1}{3^2}} - \underset{n=4}{\frac{1}{4^2}} + \dots$

$\boxed{n^2}$

b) $\sum_{n=0}^{\infty} (-1)^{n}\left(\frac{1}{*}\right) = \underset{n=0}{\frac{1}{1^2}} - \underset{n=1}{\frac{1}{2^2}} + \underset{n=2}{\frac{1}{3^2}} - \underset{n=3}{\frac{1}{4^2}} + \dots$

$\boxed{(n+1)^2}$

c) $\sum_{n=*}^{\infty} (-1)^{n}\left(\frac{-1}{(n-2)^2}\right) = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

$(n-2)^2 = 1^2$, so $n-2 = 1$, so $n = \boxed{3}$

3. $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1}$: $a_1 = \left(\frac{-1}{2}\right)^{1-1} = \left(\frac{-1}{2}\right)^{0} = 1$

$a_2 = \left(\frac{-1}{2}\right)^{2-1} = \left(\frac{-1}{2}\right)^{1} = \frac{-1}{2}$

$a_3 = \left(\frac{-1}{2}\right)^{3-1} = \left(\frac{-1}{2}\right)^{2} = \frac{1}{4}$

$\sum_{n=1}^{\infty} -\left(\frac{1}{2}\right)^{n-1}$: $a_1 = -\left(\frac{1}{2}\right)^{1-1} = -\left(\frac{1}{2}\right)^{0} = -1 \rightarrow$ $\boxed{\text{Different}}$

5. $\sum_{n=0}^{\infty} (-1)^n\left(\frac{1}{2}\right)^n$: $a_0 = (-1)^0\left(\frac{1}{2}\right)^0 = 1 \cdot 1 = 1$ ✓

$a_1 = (-1)^1\left(\frac{1}{2}\right)^1 = -1 \cdot \frac{1}{2} = \frac{-1}{2}$ ✓

$a_2 = (-1)^2\left(\frac{1}{2}\right)^2 = 1 \cdot \frac{1}{4} = \frac{1}{4}$ ✓

$\boxed{\text{Same}}$

7. $1 + 1.1 + 1.11 + 1.111 + 1.1111 + \dots$

Partial sums : 1, 2.1, 3.21, 4.321, 5.4321, ... $\rightarrow$ $\boxed{\text{Diverges}}$

9. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} + \dots$

Partial sums : 0.5, 0.75, 0.875, 0.9375, 0.96875, 0.984375, ... $\boxed{\text{Converges to 1}}$

11. $1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots + \left(\frac{2}{3}\right)^n + \dots$

$a_0 = 1$, $r = 2/3$

$S = \frac{a_0}{1-r} = \frac{1}{1-2/3} = \frac{1}{1/3} = \boxed{3}$ (converges)

13. $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)\left(\frac{2}{3}\right)^n \quad a_0 = \frac{5}{4}\cdot\left(\frac{2}{3}\right)^0 = \frac{5}{4}\cdot 1 = \frac{5}{4}, \quad r = \frac{2}{3}$

$S = \frac{a_0}{1-r} = \frac{5/4}{1-2/3} = \frac{5/4}{1/3} = \frac{5}{4}\cdot\frac{3}{1} = \boxed{\frac{15}{4}}$ (converges)

15. $\sum_{n=0}^{\infty} \cos(n\pi) = \cos 0 + \cos\pi + \cos 2\pi + \cos 3\pi + \cos 4\pi + \cos 5\pi + \dots$

$= 1 + -1 + 1 + -1 + 1 + -1 + \dots$

Partial sums: $1, 0, 1, 0, 1, 0, \dots \rightarrow$ $\boxed{\text{Diverges}}$

17. $\sum_{n=0}^{\infty} \left(\sin\left(\frac{\pi}{4}+n\pi\right)\right)^n = \left(\sin\left(\frac{\pi}{4}+0\pi\right)\right)^0 + \left(\sin\left(\frac{\pi}{4}+1\pi\right)\right)^1 + \left(\sin\left(\frac{\pi}{4}+2\pi\right)\right)^2 + \left(\sin\left(\frac{\pi}{4}+3\pi\right)\right)^3 + \dots$

$= 1 + \left(\sin\frac{5\pi}{4}\right)^1 + \left(\sin\frac{\pi}{4}\right)^2 + \left(\sin\frac{5\pi}{4}\right)^3 + \dots$

$= 1 + \left(\frac{-\sqrt{2}}{2}\right)^1 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^3 + \left(\frac{\sqrt{2}}{2}\right)^4 + \dots$

$= 1 - \left(\frac{\sqrt{2}}{2}\right)^1 + \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^3 + \left(\frac{\sqrt{2}}{2}\right)^4 - \dots$

$\times -\frac{\sqrt{2}}{2} \quad \times -\frac{\sqrt{2}}{2} \quad \times -\frac{\sqrt{2}}{2} \quad \times -\frac{\sqrt{2}}{2} \qquad r = -\frac{\sqrt{2}}{2}, \; a_0 = 1$

$S = \frac{a_0}{1-r} = \frac{1}{1 - -\frac{\sqrt{2}}{2}} = \frac{1}{1+\frac{\sqrt{2}}{2}} = \frac{1}{\frac{2}{2}+\frac{\sqrt{2}}{2}} = \frac{1}{\frac{2+\sqrt{2}}{2}} = \boxed{\frac{2}{2+\sqrt{2}}}$ (converges)

19. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n \quad a_1 = \left(\frac{e}{\pi}\right)^1 = \frac{e}{\pi}, \quad r = \frac{e}{\pi}$ (converges)

$S = \frac{a_1}{1-r} = \frac{e/\pi}{1-\frac{e}{\pi}} = \frac{e/\pi}{\frac{\pi}{\pi}-\frac{e}{\pi}} = \frac{e/\pi}{\frac{\pi-e}{\pi}} = \frac{e}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\pi-e} = \boxed{\frac{e}{\pi-e}}$

21. $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n \quad a_0 = (2x)^0 = 1$
$r = 2x$

$|2x| < 1$, so $2x < 1$ and $2x > -1$ $\quad \boxed{(-1/2, 1/2)}$
$x < 1/2 \qquad x > -1/2$

$S = \frac{a_0}{1-r} = \boxed{\frac{1}{1-2x}}$

23. $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-3)^n = \sum_{n=0}^{\infty} \left(\frac{-1}{2}(x-3)\right)^n = \sum_{n=0}^{\infty} \left(\frac{-1}{2}x+\frac{3}{2}\right)^n$ $\quad a_0 = \left(\frac{-1}{2}x+\frac{3}{2}\right)^0 = 1$

$r = \frac{-1}{2}x + \frac{3}{2}$

$\left|-\frac{1}{2}x+\frac{3}{2}\right| < 1$, so $-\frac{1}{2}x + \frac{3}{2} < 1$ and $\frac{-1}{2}x + \frac{3}{2} > -1$

$-\frac{1}{2}x < -\frac{1}{2}$ $\qquad -\frac{1}{2}x > \frac{-5}{2}$

$x > 1$ $\qquad x < 5$ $\qquad \boxed{(1,5)}$

$$S = \frac{a_0}{1-r} = \frac{1}{1-\left(\frac{-1}{2}x+\frac{3}{2}\right)} = \frac{1}{1+\frac{1}{2}x-\frac{3}{2}} = \frac{1}{\frac{1}{2}x-\frac{1}{2}} = \frac{1}{\frac{1}{2}(x-1)} = \boxed{\frac{2}{x-1}}$$

25. $\sum_{n=0}^{\infty} (\sin x)^n$ $\quad a_0 = (\sin x)^0 = 1$

$r = \sin x$

$|\sin x| < 1$, so $-1 < \sin x < 1$

$x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc. so avoid $x = \frac{\pi}{2} + \pi$

IOC: $(-\pi/2, \pi/2)$ so $\boxed{\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)}$

$+\pi \quad +\pi$

$+2\pi \quad +2\pi$

etc. $\quad$ etc.

$$S = \frac{a_0}{1-r} = \boxed{\frac{1}{1-\sin x}}$$