Section 10.1: 37–45 e.o.o, 51–61 odd

37. $\sum_{n=0}^{\infty} x^n = 20$ $\quad a_0 = x^0 = 1$, $r = x$

$$S = \frac{a_0}{1-r} = \frac{1}{1-x} \rightarrow \frac{1}{1-x} = 20 \rightarrow 1 = 20(1-x)$$

$$1 = 20 - 20x \rightarrow 20x = 19 \rightarrow \boxed{x = \frac{19}{20}}$$

41. $0.\overline{234}$

$$0.234 + 0.000234 + 0.000000234 + \dots$$

$$\frac{234}{10^3} + \frac{234}{10^3}\left(\frac{1}{10^3}\right)^1 + \frac{234}{10^3}\left(\frac{1}{10^3}\right)^2 + \dots$$

$$\sum_{n=0}^{\infty} \frac{234}{10^3}\left(\frac{1}{10^3}\right)^n \qquad a_0 = 234/10^3, \quad r = 1/10^3$$

$$S = \frac{a_0}{1-r} = \frac{234/10^3}{1-\frac{1}{10^3}} = \frac{234/10^3}{\frac{10^3}{10^3} - \frac{1}{10^3}} = \frac{234/10^3}{999/10^3} = \frac{234}{\cancel{10^3}} \cdot \frac{\cancel{10^3}}{999} = \frac{234 \div 9}{999 \div 9} = \boxed{\frac{26}{111}}$$

45. $1.\overline{414}$

$$1 + 0.414 + 0.000414 + 0.000000414 + \dots$$

$$1 + \frac{414}{10^3} + \frac{414}{10^3}\left(\frac{1}{10^3}\right)^1 + \frac{414}{10^3}\left(\frac{1}{10^3}\right)^2 + \dots$$

$$1 + \sum_{n=0}^{\infty} \frac{414}{10^3}\left(\frac{1}{10^3}\right)^n \qquad a_0 = 414/10^3, \quad r = 1/10^3$$

$$S = 1 + \frac{a_0}{1-r} = 1 + \frac{414/10^3}{1 - 1/10^3} = \frac{414/10^3}{999/10^3} + 1 = \frac{414}{999} + \frac{999}{999} = \frac{1413 \div 9}{999 \div 9} = \boxed{\frac{157}{111}}$$

51. Each semicircle: $\frac{\pi r^2}{2}$ with $r = \frac{1}{2^n}$ $\rightarrow \frac{\pi}{2}\left(\frac{1}{2^n}\right)^2 = \frac{\pi}{2} \cdot \frac{1}{2^n} \cdot \frac{1}{2^n}$

2^n semicircles per row, so each row's area: $\cancel{2^n} \cdot \frac{\pi}{2} \cdot \frac{1}{\cancel{2^n}} \cdot \frac{1}{2^n} = \frac{\pi}{2} \cdot \frac{1}{2^n}$

Sum of area of all rows: $\sum_{n=1}^{\infty} \frac{\pi}{2}\left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{\pi}{2}\left(\frac{1}{2}\right)^n$

$$a_1 = \frac{\pi}{2}\left(\frac{1}{2}\right)^1 = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}, \quad r = \frac{1}{2}$$

$$S = \frac{a_1}{1-r} = \frac{\pi/4}{1-1/2} = \frac{\pi/4}{1/2} = \frac{\pi}{4} \cdot \frac{2}{1} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

53. $\sum_{n=1}^{\infty} ar^{n-1}$ $\quad a_1 = a \cdot r^{1-1} = a \cdot r^0 = a \cdot 1 = a$ (initial term)

r (common ratio)

From question 52, $S = \dfrac{a - ar^n}{1-r}$.

When $|r|<1$: $\lim_{n\to\infty} \dfrac{a-ar^n}{1-r} = \dfrac{a(1-r^\infty)}{1-r} = \dfrac{a(1-0)}{1-r} = \boxed{\dfrac{a}{1-r}}$

When $|r|>1$: $\lim_{n\to\infty} \dfrac{a-ar^n}{1-r} = \dfrac{a(1-r^\infty)}{1-r} = \dfrac{a(1-\infty)}{1-r} = \left(\dfrac{a}{1-r}\right)(-\infty)$

Constant $\times$ Infinity $\rightarrow$ $\boxed{\text{Diverges}}$

55. $\dfrac{x}{1-2x} \rightarrow$ $\quad a_0 = x$, $\; r = 2x$

$$\boxed{x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \dots + x(2x)^n = 2^n x^{n+1}}$$

$2x < 1$ and $2x > -1$

$x < 1/2$ and $x > -1/2$ $\rightarrow$ IOC: $\boxed{(-1/2, 1/2)}$

57. $\dfrac{1}{1+(x-4)} = \dfrac{1}{1--(x-4)} \rightarrow$ $\quad a_0 = 1$, $\; r = -(x-4)$

$$\boxed{1 - (x-4) + (x-4)^2 - (x-4)^3 + (x-4)^4 - \dots + (-(x-4))^n = (-1)^n (x-4)^n}$$

$-(x-4) < 1$ and $-(x-4) > -1$

$-x+4 < 1 \qquad -x+4 > -1$

$-x < -3 \qquad -x > -5$

$x > 3 \qquad x < 5$ $\rightarrow$ IOC: $\boxed{(3,5)}$

59. $\frac{1}{2-x} = \frac{1}{1-(x-1)} \rightarrow \begin{matrix} a_0 = 1 \\ r = x-1 \end{matrix}$

$$\boxed{1 + (x-1) + (x-1)^2 + (x-1)^3 + \dots + (x-1)^n}$$

$x-1 < 1$ and $x-1 > -1$

$x < 2$ $\qquad$ $x > 0 \rightarrow$ IOC: $\boxed{(0,2)}$

61. $\sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n$ for $t \neq 0$

a) $t = 1$

$\sum_{n=0}^{\infty} \left(\frac{1}{1+1}\right)^n = \left(\frac{1}{2}\right)^n \rightarrow \begin{matrix} a_0 = (1/2)^0 = 1 \\ r = 1/2 \end{matrix}$

$S = \frac{a_0}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = \boxed{2}$

b) $\frac{t}{1+t} < 1$ and $\frac{t}{1+t} > -1$

$t < 1+\cancel{t}$ $\qquad$ $t > -1-t$

$-t \quad -\cancel{t}$ $\qquad$ $2t > -1$

$0 < 1$ $\qquad$ $\boxed{t > -1/2}$

True for any t

c) $S = \frac{a_0}{1-r} = \frac{1}{1-\frac{t}{1+t}} = \frac{1}{\frac{1+\cancel{t}-\cancel{t}}{1+t}} = \frac{1}{\frac{1}{1+t}} = 1+t > 10$

$\boxed{t > 9}$