Power Series (Section 10.1)

* Infinite Series: expressed in the form $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ "sum of"

a_n is the "n^{th} term" in the series.

* Partial Sums: $\sum_{k=1}^{n} a_k$ would sum the series from 1 to n.

* Converge / Diverge: a series converges if a limit as $n \to \infty$ exists.
  a series diverges if there is no limit.

* Geometric Series: $\sum_{n=1}^{\infty} a \cdot r^{n-1}$

if $|r| < 1 \to$ the $\boxed{S = \frac{a}{1-r}}$ converges

if $|r| \geq 1 \to$ the series diverges

* The interval of convergence is $-1 < r < 1$

ex: $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$ conv or div?

$a_1 = 3\left(\frac{1}{2}\right)^0 = 3$

$a_2 = 3\left(\frac{1}{2}\right)^1 = \frac{3}{2}$

$\vdots$

$\left|\frac{1}{2}\right| < 1$ $\boxed{\therefore \text{ converge}}$

$S = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = \boxed{6}$

ex: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \left(-\frac{1}{2}\right)^{n-1}$

conv or div?

$\left|-\frac{1}{2}\right| < 1$ $\boxed{\therefore \text{ Converge}}$

$S = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}}$

$\boxed{S = \frac{2}{3}}$

ex: $\frac{\pi}{2} + \frac{\pi^2}{4} + \frac{\pi^3}{8} + \ldots$ conv or div?

$r = \frac{\pi}{2}$

$\left|\frac{\pi}{2}\right| \geq 1$ $\boxed{\therefore \text{ diverges}}$

* Power Series: $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$

This is centered @ $x = 0$

general form

* If centered @ $x=a \rightarrow \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

$\ldots + c_n(x-a)^n$

ex: $\boxed{\sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + x^3 + \ldots + x^n = \frac{1}{1-x}$ ✓

1) what would $\frac{x}{1-x}$ look like?

$x\left(\frac{1}{1-x}\right) = x\left[1 + x + x^2 + x^3 + \ldots + x^n\right]$

$= x + x^2 + x^3 + x^4 + \ldots + x^{n+1} = \sum_{n=0}^{\infty} x^{n+1}$

2) what about $\frac{1}{1-2x}$

$= 1 + (2x) + (2x)^2 + (2x)^3 + \ldots + (2x)^n$

$= 1 + 2x + 4x^2 + 8x^3 + \ldots + (2x)^n$

$= \sum_{n=0}^{\infty} (2x)^n$

ex: find a power series for $\frac{1}{(1-x)^2}$. $\rightarrow$ How does $\frac{1}{(1-x)^2}$ relate to $\frac{1}{1-x}$?

$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[1 + x + x^2 + x^3 + \ldots + x^n\right]$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} + \ldots = \sum_{n=0}^{\infty} nx^{n-1}$

ex: find a power series for $\ln(1+x)$ ✓ $\rightarrow$ How does this relate to $\frac{1}{1+x}$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n$

* $\ln(1+x) = \int \frac{1}{1+x} = \int 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

* Repeating Decimals : (*Section 10.1)

Steps ① Set the decimal $= x$.

② Multiply both sides by a factor of 10 that moves the repeating part to the left side of the decimal.

③ Subtract these two equations.

④ Solve for x & reduce the fraction.

ex: $[0.\overline{53} = x]\,100$

$$\begin{aligned} 53.\overline{53} &= 100x \\ -0.\overline{53} &= -x \\ \hline 53 &= 99x \\ \frac{53}{99} &= \frac{99x}{99} \end{aligned}$$

$$\boxed{x = \frac{53}{99}}$$

\#45 $[1.\overline{414} = x]\,1000$

$$\begin{aligned} 1414.\overline{414} &= 1000x \\ -1.\overline{414} &= -x \\ \hline 1413 &= 999x \\ \frac{1413}{999} &= \frac{999x}{999} \end{aligned}$$

$$x = \frac{1413}{999} = \boxed{\frac{157}{111}}$$

Using series: $1.\overline{414} = 1 + .414 + .000414 + .000000414 + \ldots$

$$= 1 + \left[\frac{414}{1000} + \frac{414}{10^6} + \frac{414}{10^9} + \ldots\right]$$

$*1/10^3$ $\therefore r = \frac{1}{10^3}$

$a_1 = \frac{414}{10^3}$

$$S = \frac{\frac{414}{10^3}}{1 - \frac{1}{10^3}} = \frac{414}{10^3 - 1} = \frac{414}{999}$$

$$= 1 + \left[\frac{414}{999}\right]$$

$$= \frac{999 + 414}{999} = \frac{1413}{999} = \frac{157}{111} \checkmark$$