Section 10.2: QR - 1-10all, Ex - 5, 9, 15, 18, 23, 27

Quick Review Problems:

1. $e^{2x} \to 2e^{2x} \to 4e^{2x} \to 8e^{2x} \to 16e^{2x} \ldots \boxed{2^n e^{2x}}$

   $n=1 \quad n=2 \quad n=3 \quad n=4$

2. $\dfrac{1}{x-1} = (x-1)^{-1} \to -(x-1)^{-2} \to 2(x-1)^{-3} \to -6(x-1)^{-4} \to 24(x-1)^{-5} \ldots$

   $n=1 \quad n=2 \quad n=3 \quad n=4$

   $\boxed{(-1)^n n!\,(x-1)^{-(n+1)}}$

3. $3^x \to 3^x \cdot \ln 3 \to 3^x(\ln 3)^2 \to 3^x(\ln 3)^3 \to 3^x(\ln 3)^4 \ldots \boxed{3^x(\ln 3)^n}$

   $n=1 \quad n=2 \quad n=3 \quad n=4$

4. $\ln x \to \dfrac{1}{x} = x^{-1} \to -x^{-2} \to 2x^{-3} \to -6x^{-4} \to 24x^{-5} \ldots \boxed{(-1)^{n-1}(n-1)!\,x^{-n}}$

   $n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

5. $x^n \to nx^{n-1} \to n(n-1)x^{n-2} \to n(n-1)(n-2)x^{n-3} \ldots$

   x^1 (1 derivative) $= 1$

   x^2 (2 derivatives) $\to 2x \to 2$

   x^3 (3 derivatives) $\to 3x^2 \to 6x \to 6$

   x^4 (4 derivatives) $\to 4x^3 \to 12x^2 \to 24x \to 24$

   n^{th} derivative of $x^n = \boxed{n!}$

6. $y = \dfrac{x^n}{n!} \to \dfrac{dy}{dx} = \dfrac{nx^{n-1}}{n!} = \boxed{\dfrac{x^{n-1}}{(n-1)!}}$

7. $y = \dfrac{2^n(x-a)^n}{n!} \to \dfrac{dy}{dx} = \dfrac{2^n \cdot n(x-a)^{n-1}}{n!} = \boxed{\dfrac{2^n(x-a)^{n-1}}{(n-1)!}}$

8. $y = (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \to \dfrac{dy}{dx} = \dfrac{(-1)^n(2n+1)x^{2n}}{(2n+1)!} = \boxed{\dfrac{(-1)^n x^{2n}}{(2n)!}}$

9. $y = \dfrac{(x+a)^{2n}}{(2n)!} \to \dfrac{dy}{dx} = \dfrac{2n(x+a)^{2n-1}}{(2n)!} = \boxed{\dfrac{(x+a)^{2n-1}}{(2n-1)!}}$

10. $y=\frac{(1-x)^n}{n!} \rightarrow \frac{dy}{dx}=\frac{n(1-x)^{n-1}(-1)}{n!}=\boxed{\frac{-(1-x)^{n-1}}{(n-1)!}}$

Exercises:

5. $\sin 2x$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 2x = 2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\dots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}$$

$$\sin 2x = 2x-\frac{8x^3}{6}+\frac{32x^5}{120}-\frac{128x^7}{5040}+\dots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \boxed{2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\frac{8x^7}{315}+\dots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}}$$

Converges for all real x.

9. $\cos(x/2)$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots+\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(\tfrac{1}{2}x) = 1-\frac{(\frac{1}{2}x)^2}{2!}+\frac{(\frac{1}{2}x)^4}{4!}-\frac{(\frac{1}{2}x)^6}{6!}+\dots+\frac{(-1)^n(\frac{1}{2}x)^{2n}}{(2n)!}$$

$$\cos(\tfrac{1}{2}x) = 1-\frac{\frac{1}{4}x^2}{2}+\frac{\frac{1}{16}x^4}{24}-\frac{\frac{1}{64}x^6}{720}+\dots+\frac{(-1)^n(\frac{1}{2}x)^{2n}}{(2n)!}$$

$$\cos(\tfrac{1}{2}x) = \boxed{1-\frac{1}{8}x^2+\frac{1}{384}x^4-\frac{1}{46{,}080}x^6+\dots+\frac{(-1)^n(\frac{1}{2}x)^{2n}}{(2n)!}}$$

Converges for all real x.

15. $f(x) = x^3 - 2x + 4$

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

$f'''(x) = 6$

a) $x = 0$

$f(0) = 4,\ f'(0) = -2,\ f''(0) = 0,\ f'''(0) = 6$

$$4 + \frac{-2x}{1!} + \frac{0x^2}{2!} + \frac{6x^3}{3!} = 4 - 2x + 0x^2 + x^3 = \boxed{4 - 2x + x^3}$$

b) $x = 1$

$f(1) = 3,\ f'(1) = 1,\ f''(1) = 6,\ f'''(1) = 6$

$$3 + \frac{1(x-1)}{1!} + \frac{6(x-1)^2}{2!} + \frac{6(x-1)^3}{3!} = \boxed{3 + (x-1) + 3(x-1)^2 + (x-1)^3}$$

18. $f(x) = \frac{1}{x} = x^{-1},\ a = 2$

$f'(x) = -x^{-2} = \frac{-1}{x^2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

$f'''(x) = -6x^{-4} = \frac{-6}{x^4}$

$f(2) = 1/2$

$f'(2) = -1/4$

$f''(2) = 2/8 = 1/4$

$f'''(2) = -6/16 = -3/8$

Order 0 : $1/2$

Order 1 : $\frac{1}{2} + \frac{-\frac{1}{4}(x-2)}{1!} = \frac{1}{2} - \frac{1}{4}(x-2)$

Order 2 : $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1/4(x-2)^2}{2!} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$

Order 3 : $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + \frac{-3/8(x-2)^3}{3!} = \boxed{\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{3}{48}(x-2)^3}$

23. $f(1)=4,\ f'(1)=-1,\ f''(1)=3,\ f'''(1)=2$

a) $4+\frac{-1(x-1)}{1!}+\frac{3(x-1)^2}{2!}+\frac{2(x-1)^3}{3!} = \boxed{4-(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3}$

$f(1.2) \approx 4-(1.2-1)+\frac{3}{2}(1.2-1)^2+\frac{1}{3}(1.2-1)^3 = \boxed{3.863}$

b) $4-(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3$

Take the derivative

$\boxed{-1+3(x-1)+(x-1)^2}$

$f'(1.2) \approx -1+3(1.2-1)+(1.2-1)^2 = \boxed{-0.36}$

27. $f(x)=\sqrt{1+x}=(1+x)^{1/2}$ $\qquad f(0)=1$

a) $f'(x)=\frac{1}{2}(1+x)^{-1/2}$ $\qquad f'(0)=1/2$

$f''(x)=\frac{-1}{4}(1+x)^{-3/2}$ $\qquad f''(0)=-1/4$

$f'''(x)=\frac{3}{8}(1+x)^{-5/2}$ $\qquad f'''(0)=3/8$

$1+\frac{1/2\,x}{1!}\ \frac{-1/4\,x^2}{2!}+\frac{3/8\,x^3}{3!} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{3}{48}x^3 = \boxed{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3}$

b) For $g(x)=\sqrt{1+x^2}$, substitute x^2 for each x from part a.

$1+\frac{1}{2}(x^2)-\frac{1}{8}(x^2)^2+\frac{1}{16}(x^2)^3 = \boxed{1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6}$

c) $h(x)$ is the antiderivative of $\sqrt{1+x^2}$, and $h(0)=5$.

$$\int\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6\right)dx = x+\frac{1}{2}\cdot\frac{1}{3}x^3-\frac{1}{8}\cdot\frac{1}{5}x^5+\frac{1}{16}\cdot\frac{1}{7}x^7+C$$

$h(x) = x+\frac{1}{6}x^3-\frac{1}{40}x^5+\frac{1}{112}x^7+C$

$5 = 0+0+0+0+C \rightarrow \quad C=5$

$h(x) = x+\frac{1}{6}x^3-\frac{1}{40}x^5+\frac{1}{112}x^7+5$

First 4 terms: $\boxed{5+x+\frac{1}{6}x^3-\frac{1}{40}x^5}$