Section 10.2: 7, 11, 17, 20, 25, 29–35 all

7. $\tan^{-1} x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots + (x^2)^n(-1)^n = (-1)^n x^{2n}$$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\tan^{-1} x^2 = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \dots + \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$$

$$\tan^{-1} x^2 = \boxed{x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \dots + \frac{(-1)^n x^{4n+2}}{2n+1}} \text{ (for } |x| \le 1)$$

11. $\frac{x}{1-x^3}$: $a_0 = x$, $r = x^3$

$$\frac{x}{1-x^3} = \boxed{x + x^4 + x^7 + \dots + x(x^3)^n = x^1 x^{3n} = x^{3n+1}} \text{ (for } |x^3| < 1, \text{ so } -1 < x < 1)$$

17.
$f(x) = x^4$ $\quad f(0) = 0$ $\quad f(1) = 1$

$f'(x) = 4x^3$ $\quad f'(0) = 0$ $\quad f'(1) = 4$

$f''(x) = 12x^2$ $\quad f''(0) = 0$ $\quad f''(1) = 12$

$f'''(x) = 24x$ $\quad f'''(0) = 0$ $\quad f'''(1) = 24$

$$x=0:\ 0 + \frac{0x}{1!} + \frac{0x^2}{2!} + \frac{0x^3}{3!} = \boxed{0}$$

$$x=1:\ 1 + \frac{4(x-1)}{1!} + \frac{12(x-1)^2}{2!} + \frac{24(x-1)^3}{3!} = \boxed{1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3}$$

20.
$f(x) = \cos x$, $a = \pi/4$ $\quad f(\pi/4) = \sqrt{2}/2$

$f'(x) = -\sin x$ $\quad f'(\pi/4) = -\sqrt{2}/2$

$f''(x) = -\cos x$ $\quad f''(\pi/4) = -\sqrt{2}/2$

$f'''(x) = \sin x$ $\quad f'''(\pi/4) = \sqrt{2}/2$

Order 0: $\boxed{\frac{\sqrt{2}}{2}}$

Order 1: $\frac{\sqrt{2}}{2} + \frac{-\sqrt{2}/2\,(x - \pi/4)}{1!} = \boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \pi/4)}$

Order 2: $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \pi/4) + \frac{-\sqrt{2}/2\,(x-\pi/4)^2}{2!} = \boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{4}(x - \pi/4)^2}$

20. (continued)

Order 3: $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\pi/4) - \frac{\sqrt{2}}{4}(x-\pi/4)^2 + \frac{\sqrt{2}/2\,(x-\pi/4)^3}{3!}$

$$\boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\pi/4) - \frac{\sqrt{2}}{4}(x-\pi/4)^2 + \frac{\sqrt{2}}{12}(x-\pi/4)^3}$$

25. $e^{x/2} = e^{\frac{1}{2}x}$

a) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

$$e^{\frac{1}{2}x} = 1 + \tfrac{1}{2}x + \frac{(\frac{1}{2}x)^2}{2!} + \cdots + \frac{(\frac{1}{2}x)^n}{n!}$$

$$e^{\frac{1}{2}x} = \boxed{1 + \frac{1}{2}x + \frac{1}{8}x^2 + \cdots + \frac{x^n}{2^n \cdot n!}}$$

b) $g(x) = \frac{e^x - 1}{x} = \frac{1}{x}(e^x - 1)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$e^x - 1 = \cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} - \cancel{1}$$

$$\frac{1}{x}(e^x - 1) = \frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) = \boxed{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{n-1}}{n!}}$$

c) $g(x) = \frac{e^x - 1}{x}$, so $g'(x) = \frac{x(e^x) - (e^x - 1)1}{x^2} = \frac{xe^x - e^x + 1}{x^2}$

$$g'(1) = \frac{1e^1 - e^1 + 1}{1^2} = \frac{\cancel{e} - \cancel{e} + 1}{1} = \frac{1}{1} = 1$$

From part b, $g(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{n-1}}{n!} + \frac{x^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$

$$g'(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \cdots + \frac{nx^{n-1}}{(n+1)!} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}$$

$$g'(1) = \sum_{n=1}^{\infty} \frac{n \cdot \cancel{1^{n-1}}^{(1)}}{(n+1)!} = \boxed{\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1}$$ since $g'(1) = 1$ from the derivative.

29. cos18 accurate within 0.001

cos18 ≈ 0.6603

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Type: 0 → N: 1 → T Enter

N+1 → N: T+(-1)^N · 18^(2N)/(2N)! → T Enter

Continue pressing enter until the value is with 0.001 of 0.6603

This happens after 26 repetitions.

We started at $n=0$, so $\boxed{\text{27 terms}}$ total.

30. A Taylor polynomial of any order has a finite number of terms.

sinx is perfectly represented by the infinite series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

A polynomial stops at a certain order (degree), so there will be some amount of error between a Taylor polynomial and sinx.

31. sinx is an odd function, so its series has all odd powers and odd factorials. cosx is an even function, so its series has all even powers and even factorials.

32. $\frac{f^5(x)\,x^5}{5!}$

$f(x) = \sin 3x$

$f'(x) = 3\cos 3x$

$f''(x) = -9\sin 3x$

$f^3(x) = -27\cos 3x$

$f^4(x) = 81\sin 3x$

$f^5(x) = 243\cos 3x$

$f^5(0) = 243\cos 0 = 243 \cdot 1 = 243$

$$\frac{f^5(0)\,x^5}{5!} = \frac{243x^5}{120}$$

$$\frac{243}{120} = \boxed{\frac{81}{40}}$$

33. $\frac{f'''(x)(x-2)^3}{3!}$

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3} = \frac{2}{x^3}$

$f'''(2) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$

$$\frac{f'''(2)(x-2)^3}{3!} = \frac{\tfrac{1}{4}(x-2)^3}{6}$$

$$\frac{1/4}{6} = \frac{1}{4} \cdot \frac{1}{6} = \boxed{\frac{1}{24}}$$

34. Linearization uses a tangent line to approximate values on a curve near a particular x-value. Taylor polynomials use a series of terms to approximate values on a curve more accurately. Taylor polynomials use constant, linear, quadratic, cubic, etc. terms instead of stopping at only linear.

35. $f(x) = \dfrac{4x}{x^2+1}$

a) $f'(x) = \dfrac{(x^2+1)4 - (4x)2x}{(x^2+1)^2} = \dfrac{4x^2+4-8x^2}{(x^2+1)^2} = \dfrac{4-4x^2}{(x^2+1)^2}$

$f(0) = \dfrac{0}{0+1} = \dfrac{0}{1} = 0$ $\qquad f'(\sqrt{3}) = \dfrac{4-4(3)}{(3+1)^2} = \dfrac{4-12}{4^2} = \dfrac{-8}{16} = \dfrac{-1}{2}$

$f'(0) = \dfrac{4-0}{(0+1)^2} = \dfrac{4}{1} = 4$ $\qquad f(\sqrt{3}) = \dfrac{4\sqrt{3}}{3+1} = \dfrac{\cancel{4}\sqrt{3}}{\cancel{4}} = \sqrt{3}$

$x=0: \ 0 + \dfrac{4x}{1!} = 0 + 4x = \boxed{4x}$

$x=\sqrt{3}: \ \sqrt{3} + \dfrac{-\frac{1}{2}(x-\sqrt{3})}{1!} = \boxed{\sqrt{3} - \frac{1}{2}(x-\sqrt{3})}$

b) If $x=a$ is an inflection point, then $f''(a) = 0$. Therefore, the coefficient of the quadratic term in the Taylor polynomial is zero.

$$f(a) + \frac{f'(a)(x-a)}{1!} + \frac{\overbrace{f''(a)}^{0}(x-a)^2}{2!} = f(a) + f'(a)(x-a) + \cancel{0}$$

The constant term and linear term are a good approximation to the curve near an inflection point.