Taylor Series (Section 10.2)

* A Taylor polynomial: approximates a function using a poly.

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n$$

$$a_0 = P(0) \quad a_1 = \frac{P'(0)}{1!} \quad a_2 = \frac{P''(0)}{2!} \quad a_3 = \frac{P'''(0)}{3!} \quad a_4 = \frac{P^4(0)}{4!} \quad \dots \quad a_n = \frac{P^n(0)}{n!}$$

ex: $\ln(1+x)$ Find a 4^{th} degree poly.

$$P(0) = \ln(1+0) = 0$$

$$P'(x) = \frac{1}{1+x} \rightarrow P'(0) = \frac{1}{1+0} = 1$$

$$P''(x) = \frac{-1}{(1+x)^2} \rightarrow P''(0) = \frac{-1}{(1+0)^2} = -1$$

$$P'''(x) = \frac{2}{(1+x)^3} \rightarrow P'''(0) = \frac{2}{(1+0)^3} = 2$$

$$P^4(x) = \frac{-6}{(1+x)^4} \rightarrow P^4(0) = \frac{-6}{(1+0)^4} = -6$$

$$P_4(x) = 0 + \frac{1x}{1!} - \frac{1x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

$$\boxed{P_4(x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}$$

ex: construct a 7^{th} order Taylor poly for $\sin x$.

$$P(0) = \sin 0 = 0 \qquad P^4(0) = \sin 0 = 0$$

$$P'(0) = \cos 0 = 1 \qquad P^5(0) = \cos 0 = 1$$

$$P''(0) = -\sin 0 = 0 \qquad P^6(0) = -\sin 0 = 0$$

$$P'''(0) = -\cos 0 = -1 \qquad P^7(0) = -\cos 0 = -1$$

$$P_7(x) = \cancel{0} + \frac{x}{1!} + \cancel{\frac{0x^2}{2!}} - \frac{x^3}{3!} + \cancel{\frac{0x^4}{4!}} + \frac{1x^5}{5!} + \cancel{\frac{0x^6}{6!}} - \frac{x^7}{7!}$$

$$\boxed{P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$$

* Taylor series centered @ $x=0 \rightarrow$ called a "Maclaurin Series"

$$= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^n(0)x^n}{n!} = \sum_{k=0}^{\infty} \frac{f^k(0)x^k}{k!}$$

* If not centered @ $x=0 \rightarrow$ plug $(x-a)$ in for x.

ex: Write a Taylor series centered @ $x=2$ for $f(x)=e^x$

$f(2)=e^2$

$f'(2)=e^2$

$f''(2)=e^2 \dots$

so: $e^2 + e^2x + \frac{e^2x^2}{2!} + \frac{e^2x^3}{3!} + \dots + \frac{e^2x^n}{n!}$

$= e^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!}\right)$ (for $x=0$ as center)

$\therefore$ e^x centered @ $x=2 = e^2\left[1 + (x-2) + \frac{(x-2)^2}{2!} + \dots + \frac{(x-2)^n}{n!}\right]$

$$= \sum_{k=0}^{\infty} e^2 \cdot \frac{(x-2)^k}{k!}$$

ex: Find a 3^{rd} order Taylor poly for $f(x) = 2x^3 - 3x^2 + 4x - 5$ centered @ $x=1$.

$f(1) = 2(1)^3 - 3(1)^2 + 4(1) - 5 = -2$

$f'(1) = 6(1)^2 - 6(1) + 4 = 4$

$f''(1) = 12(1) - 6 = 6$

$f'''(1) = 12$

$P_3(x) = -2 + 4(x-1) + \frac{6(x-1)^2}{2!} + \frac{12(x-1)^3}{3!}$

$$\boxed{P_3(x) = -2 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3}$$