Section 10.3 : QR: 1–10 all, Ex: 1–10 all

Quick Review:

1. $f(x) = 2\cos 3x$, $I = [-2\pi, 2\pi]$

   Min: -2
   Max: 2 $\rightarrow$ $\boxed{M=2}$

2. $f(x) = x^2 + 3$, $I = [1, 2]$

   $1^2 + 3 = 4$
   $2^2 + 3 = 7 \rightarrow \boxed{M=7}$

3. $f(x) = 2^x$, $I = [-3, 0]$

   $2^{-3} = \frac{1}{8}$
   $2^0 = 1 \rightarrow \boxed{M=1}$

4. $f(x) = \dfrac{x}{x^2+1}$, $I = [-2, 2]$

   $\dfrac{-2}{(-2)^2+1} = \dfrac{-2}{5}$ $\quad$ $\dfrac{0}{0^2+1} = 0$ $\quad$ $\dfrac{-1}{(-1)^2+1} = \dfrac{-1}{2}$

   $\dfrac{2}{2^2+1} = \dfrac{2}{5}$ $\quad$ $\dfrac{1}{1^2+1} = \dfrac{1}{2}$ $\rightarrow \boxed{M = \frac{1}{2}}$

5. $f(x) = \begin{cases} 2-x^2, & x \le 1 \\ 2x-1, & x > 1 \end{cases}$, $I = [-3, 3]$

   $2-(-3)^2 = 2-9 = -7 \rightarrow \boxed{M=7}$
   $2 - 1^2 = 2 - 1 = 1$
   $2(1) - 1 = 2 - 1 = 1$
   $2(3) - 1 = 6 - 1 = 5$

6. $\dfrac{x}{x+1}$, $a = 0$

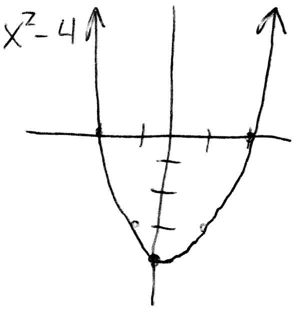
   $$f'(x) = \frac{(x+1)1 - x(1)}{(x+1)^2} = \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2} = \frac{1}{(x+1)^2} = (x+1)^{-2}$$

   $f''(x) = -2(x+1)^{-3}$
   $f'''(x) = 6(x+1)^{-4}$
   $f^4(x) = -24(x+1)^{-5}$

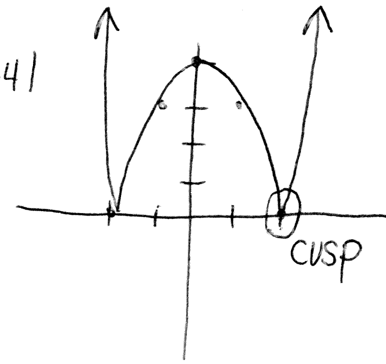
   $\boxed{\text{Yes}}$

7. $|x^2-4|$, $a=2$

x^2-4



$|x^2-4|$



cusp → $\boxed{\text{No}}$

8. $\sin x + \cos x$, $a=\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f'''(x) = -\cos x + \sin x$

$f^4(x) = \sin x + \cos x$

$\boxed{\text{Yes}}$

9. e^{-x}, $a=0$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$f'''(x) = -e^{-x}$

$f^4(x) = e^{-x}$

$\boxed{\text{Yes}}$

10. $f(x) = x^{3/2}$, $a=0$

$f'(x) = \frac{3}{2}x^{1/2} \rightarrow f'(0) = 0$

$f''(x) = \frac{3}{4}x^{-1/2} = \frac{3}{4\sqrt{x}} \rightarrow f''(0) = \frac{3}{4\sqrt{0}} \rightarrow$ Undefined $\rightarrow \boxed{\text{No}}$

Exercises:

1. e^{-2x}

$f'(x) = -2e^{-2x}$ $\quad f(0) = 1$

$f''(x) = 4e^{-2x}$ $\quad f'(0) = -2$

$f'''(x) = -8e^{-2x}$ $\quad f''(0) = 4$

$f^4(x) = 16e^{-2x}$ $\quad f'''(0) = -8$

$f^4(0) = 16$

$$1 - \frac{2x}{1!} + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!}$$

$$\boxed{1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4}$$

$f(0.2) \approx \boxed{0.6704}$

2. $\cos\left(\frac{\pi}{2}x\right)$ $\quad f(0) = 1$

$f'(x) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)$ $\quad f'(0) = 0$

$f''(x) = -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)$ $\quad f''(0) = -\frac{\pi^2}{4}$

$f'''(x) = \frac{\pi^3}{8}\sin\left(\frac{\pi}{2}x\right)$ $\quad f'''(0) = 0$

$f^4(x) = \frac{\pi^4}{16}\cos\left(\frac{\pi}{2}x\right)$ $\quad f^4(0) = \frac{\pi^4}{16}$

$$1 + \frac{0x}{1!} - \frac{\pi^2/4\, x^2}{2!} + \frac{0x^3}{3!} + \frac{\pi^4/16\, x^4}{4!}$$

$$\boxed{1 - \frac{\pi^2}{8}x^2 + \frac{\pi^4}{384}x^4}$$

$f(0.2) \approx \boxed{0.951}$

3. $5\sin(-x)$ $\quad f(0) = 0$

$f'(x) = -5\cos(-x)$ $\quad f'(0) = -5$

$f''(x) = -5\sin(-x)$ $\quad f''(0) = 0$

$f'''(x) = 5\cos(-x)$ $\quad f'''(0) = 5$

$f^4(x) = 5\sin(-x)$ $\quad f^4(0) = 0$

$$0 - \frac{5x}{1!} + \frac{0x^2}{2!} + \frac{5x^3}{3!} + \frac{0x^4}{4!}$$

$$\boxed{-5x + \frac{5}{6}x^3}$$

$f(0.2) \approx \boxed{-0.99\overline{3}}$

4. $\ln(1+x^2)$

$$f'(x) = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}$$

$$f''(x) = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$$f'''(x) = \frac{(1+x^2)^2(-4x) - (2-2x^2)2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{(1+x^2)(-4x) - (2-2x^2)2(2x)}{(1+x^2)^3}$$

$$= \frac{-4x - 4x^3 - (2-2x^2)4x}{(1+x^2)^3} = \frac{-4x-4x^3-8x+8x^3}{(1+x^2)^3} = \frac{-12x+4x^3}{(1+x^2)^3}$$

$$f^4(x) = \frac{(1+x^2)^3(-12+12x^2) - (-12x+4x^3)3(1+x^2)^2\cdot 2x}{(1+x^2)^6}$$

$$= \frac{(1+x^2)(-12+12x^2) - (-12x+4x^3)6x}{(1+x^2)^4} = \frac{-12+\cancel{12x^2}-\cancel{12x^2}+12x^4+72x-24x^4}{(1+x^2)^4}$$

4. (continued)

$$f^{4}(x) = \frac{-12-12x^4+72x}{(1+x^2)^4}$$

$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f'''(0) = 0, \quad f^{4}(0) = -12$

$$0 + \frac{0x}{1!} + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} = \boxed{x^2 - \frac{1}{2}x^4}$$

$f(0.2) \approx \boxed{0.0392}$

5. $(1-x)^{-2}$

$f'(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

$f''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$

$f'''(x) = -24(1-x)^{-5}(-1) = 24(1-x)^{-5}$

$f^{4}(x) = -120(1-x)^{-6}(-1) = 120(1-x)^{-6}$

$f(0) = 1$

$f'(0) = 2$

$f''(0) = 6$

$f'''(0) = 24$

$f''''(0) = 120$

$$1 + \frac{2x}{1!} + \frac{6x^2}{2!} + \frac{24x^3}{3!} + \frac{120x^4}{4!} = \boxed{1 + 2x + 3x^2 + 4x^3 + 5x^4}$$

$f(0.2) \approx \boxed{1.56}$

6. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x - x + \frac{x^3}{3!} = \boxed{\underbrace{\frac{x^5}{5!}}_{n=0} - \underbrace{\frac{x^7}{7!}}_{n=1} + \underbrace{\frac{x^9}{9!}}_{n=2} - \dots + \frac{(-1)^n x^{2n+5}}{(2n+5)!}}$$

7. xe^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$xe^x = \boxed{x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots + \frac{x^{n+1}}{n!}}$$

8. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots + \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$\cos 2x = 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \dots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$1+\cos 2x = 2 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \dots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$\frac{1}{2}(1+\cos 2x) = \boxed{1 - \frac{2^1x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \dots + \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}}$$

9. $\sin^2 x = 1 - \cos^2 x = 1 - (\frac{1}{2} + \frac{1}{2}\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$\cos 2x = 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \dots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$ (from #8 above)

$$1-\cos 2x = \underset{n=0}{\frac{2^2x^2}{2!}} - \underset{n=1}{\frac{2^4x^4}{4!}} + \underset{n=2}{\frac{2^6x^6}{6!}} - \dots + \frac{(-1)^n 2^{2n+2} x^{2n+2}}{(2n+2)!}$$

$$\frac{1}{2}(1-\cos 2x) = \boxed{\underset{n=0}{\frac{2^1x^2}{2!}} - \underset{n=1}{\frac{2^3x^4}{4!}} + \underset{n=2}{\frac{2^5x^6}{6!}} - \dots + \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+2)!}}$$

10. $\frac{x^2}{1-2x}$ $\quad a_0 = x^2$, $\quad r = 2x$

$$\boxed{x^2 + 2x^3 + 4x^4 + 8x^5 + \dots + x^2(2x)^n = x^2 \cdot 2^n x^n = 2^n x^{n+2}}$$