Section 10.3: 19-37 all

19. error $= |\sin x - (x - x^3/6)| \le 0.0005$ for x $\boxed{[-0.571, 0.571]}$

20. error $= \cos x - (1 - x^2/2)$ for $-0.5 < x < 0.5$

$x = -0.5$, error $= 0.00258256$

$x = 0.5$, error $= 0.00258256$

error $= \boxed{2.583 \times 10^{-3}}$

error is positive, so the approximation is $\boxed{\text{too small}}$

21. error $= \sin x - x$ for $-0.001 < x < 0.001$

$x = -0.001$, error $= 1.667 \times 10^{-10}$

$x = 0.001$, error $= -1.667 \times 10^{-10}$

error $= \boxed{1.667 \times 10^{-10}}$

$x < \sin x$ when $\sin x - x > 0$, so error positive, which is when $\boxed{x < 0}$

22. error $= |\sqrt{1+x} - (1 + \frac{1}{2}x)|$ for $-0.01 < x < 0.01$

$x = -0.01$, error $= 1.256 \times 10^{-5}$ $\rightarrow$ error $= \boxed{1.256 \times 10^{-5}}$

$x = 0.01$, error $= 1.244 \times 10^{-5}$

23. error $= |e^x - (1 + x + x^2/2)|$ for $-0.1 < x < 0.1$

$x = -0.1$, error $= 1.626 \times 10^{-4}$

$x = 0.1$, error $= 1.709 \times 10^{-4}$ $\rightarrow$ error $= \boxed{1.709 \times 10^{-4}}$

24. $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots + (-1)^n \frac{x^n}{n!}$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \dots + \frac{2x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{2}(e^x - e^{-x}) = \boxed{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}}$$

24. (continued)

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!}$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^n}{n!}$

$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots + \frac{2x^{2n}}{(2n)!}$

$\frac{1}{2}(e^x + e^{-x}) = \boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}}$

25. Remainder Estimation Theorem:

$|R_n(x)| = \frac{M|x|^{n+1}}{(n+1)!}$ where M = maximum error = $(n+1)^{st}$ derivative

$n+1$ represents the next unused term after the approximation

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}$ and $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!}$

Any derivatives of $\cosh x$ are $\sinh x$ and $\cosh x$ alternately.

$\sinh x$ and $\cosh x$ are bounded by $e^{|x|}$ for all x, so $M = e^{|x|}$.

$|R_n(x)| \leq \frac{e^{|x|}|x|^{n+1}}{(n+1)!}$, and for any value of x, $\lim_{n\to\infty} \frac{e^{|x|}|x|^{n+1}}{(n+1)!} = 0$.

Since the remainder approaches 0, the series converges to $\cosh x$.

26. $f(x) = f(a) + R(x)$

$R(x) = f'(c)(x-a)$

$f(x) = f(a) + f'(c)(x-a)$

At $x=b$, $f(b) = f(a) + f'(c)(b-a)$

$f(b) - f(a) = f'(c)(b-a)$

$f'(c) = \frac{f(b)-f(a)}{b-a}$ ← This is the Mean Value Theorem. Hence, MVT is a specific case within Taylor's Theorem.

27. $f(x) = \ln(\cos x)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x$ $\quad f'(0) = -0 = 0$

$f''(x) = -\sec^2 x$ $\quad f''(0) = -1$

a) $0 + \frac{0x^1}{1!} = \boxed{0}$

b) $0 + \frac{0x^1}{1!} - \frac{1x^2}{2!} = \boxed{\frac{-1}{2}x^2}$

28. $f(x) = e^{\sin x}$ $\quad f(0) = e^0 = 1$

$f'(x) = e^{\sin x} \cdot \cos x$ $\quad f'(0) = e^0 \cdot 1 = 1$

$f''(x) = e^{\sin x}(-\sin x) + \cos x \cdot e^{\sin x} \cdot \cos x$ $\quad f''(0) = 0 + 1 \cdot e^0 = 1$

a) $1 + \frac{1x^1}{1!} = \boxed{1+x}$

b) $1 + x + \frac{1x^2}{2!} = \boxed{1 + x + \frac{1}{2}x^2}$

29. $f(x) = (1-x^2)^{-1/2}$ $\quad f(0) = 1$

$f'(x) = -\cancel{\tfrac{1}{2}}(1-x^2)^{-3/2} \cdot -\cancel{2}x = \frac{x}{(1-x^2)^{3/2}}$ $\quad f'(0) = 0$

$f''(x) = \frac{(1-x^2)^{3/2} \cdot 1 - x \cdot \frac{3}{2}(1-x^2)^{1/2} \cdot -2x}{(1-x^2)^3}$ $\quad f''(0) = \frac{1-0}{1} = 1$

a) $1 + \frac{0x^1}{1!} = \boxed{1}$

b) $1 + \frac{0x^1}{1!} + \frac{1x^2}{2!} = \boxed{1 + \frac{1}{2}x^2}$

30. $f(x) = \sec x$ $\quad f(0) = 1$

$f'(x) = \sec x \cdot \tan x$ $\quad f'(0) = 1 \cdot 0 = 0$

$f''(x) = \sec x \cdot \sec^2 x + \tan x \cdot \sec x \cdot \tan x$ $\quad f''(0) = 1 + 0 = 1$

a) $1 + \frac{0x^1}{1!} = \boxed{1}$

b) $1 + \frac{1x^2}{2!} = \boxed{1 + \frac{1}{2}x^2}$

31. $f(x) = \tan x$ $\qquad f(0) = 0$

$f'(x) = (\sec x)^2$ $\qquad f'(0) = 1^2 = 1$

$f''(x) = 2\sec x \cdot \sec x \tan x$ $\qquad f''(0) = 2 \cdot 1 \cdot 0 = 0$

a) $0 + \frac{1x^1}{1!} = \boxed{x}$

b) $0 + \frac{1x^1}{1!} + \frac{0x^2}{2!} = \boxed{x}$

32. $f(x) = (1+x)^k$ $\qquad f(0) = 1^k = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad f'(0) = k \cdot 1^{k-1} = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad f''(0) = k(k-1) \cdot 1^{k-2} = k(k-1)$

$1 + \frac{kx^1}{1!} + \frac{k(k-1)x^2}{2!} = 1 + kx + \frac{k(k-1)}{2}x^2$

$k=3$: $1 + 3x + \frac{3 \cdot \cancel{2}}{\cancel{2}}x^2 = \boxed{1+3x+3x^2}$

error $= |(1+x)^3 - (1+3x+3x^2)| < 0.01$ when x is $\boxed{(-0.215, 0.215)}$

33. error $= |e^x - (1 + x + x^2/2 + x^3/6)|$ for $-0.1 \le x \le 0.1$

$x = -0.1$, error $= 4.085 \times 10^{-6}$

$x = 0.1$, error $= \boxed{4.251 \times 10^{-6}}$

34. $f(x) = (1-x)^{-1}$ $\qquad f(0) = 1^{-1} = 1$

$f'(x) = -1(1-x)^{-2}(-1) = 1(1-x)^{-2}$ $\qquad f'(0) = 1 \cdot 1^{-2} = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$ $\qquad f''(0) = 2 \cdot 1^{-3} = 2$

$f'''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$ $\qquad f'''(0) = 6 \cdot 1^{-4} = 6$

$1 + \frac{1x^1}{1!} + \frac{2x^2}{2!} + \frac{6x^3}{3!} = \boxed{1+x+x^2+x^3}$

error $= \left|\frac{1}{1-x} - (1+x+x^2+x^3)\right|$ for $-0.1 \le x \le 0.1$

$x = -0.1$, error $= 9.091 \times 10^{-5}$

$x = 0.1$, error $= \boxed{1.111 \times 10^{-4}}$

35. $\frac{dy}{dx} = e^{-x^2}$, $(0,2)$

a) No, must integrate.

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!} = \frac{dy}{dx}$$

$$y = \int e^{-x^2}dx = x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1) \cdot n!} + C$$

$(0,2)$, so $y = \boxed{x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1) \cdot n!} + 2}$

c) e^{-x^2} converges for all x to represent $\frac{dy}{dx}$, so $\int e^{-x^2}dx$ converges for all x to represent y.

36. $\frac{1}{1-x} \to \begin{matrix} a_0 = 1 \\ r = x \end{matrix} \to 1 + x + x^2 + x^3 + \dots + x^n$

a) $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x}$, so $\ln(1-x) = -\int (1 + x + x^2 + x^3 + \dots)dx$

$\ln(1-x) = \boxed{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots + \frac{(-1)x^n}{n}}$ (starting at $n=1$)

b) $\ln(1+x)$

$\frac{1}{1+x} = \frac{1}{1-(-x)} \to \begin{matrix} a_0 = 1 \\ r = -x \end{matrix} \to 1 - x + x^2 - x^3 + \dots + (-1)^n x^n$

$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$, so $\ln(1+x) = \int (1 - x + x^2 - x^3 + \dots)dx$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}x^n}{n}$ (starting at $n=1$)

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right)$$

$$\ln(1+x) - \ln(1-x) = \begin{matrix} x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \dots \\ + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \dots \end{matrix} = \boxed{2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots + \frac{2x^{2n+1}}{2n+1}}$$

37. a) Graph $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} = \boxed{\tan x}$

b) Graph $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} = \boxed{\sec x}$