Taylor's Theorem (Section 10.3)

* In a polynomial approximation, the error is the actual minus the approx.

Error = actual − approximation

* In the series exploration: error = $y_1 - y_2$

* A "truncation error" results from cutting the series down to a specified polynomial.

these are all the terms left over (this is a WHOLE OTHER series!)

ex: $f(x) = \frac{1}{1-x^2} \approx 1 + x^2 + x^4 + x^6$ Find the truncation error.

What are the next terms? $x^8 + x^{10} + x^{12} + \ldots$

$a_1 = x^8$
$r = x^2$

$$\boxed{\therefore\ S = \frac{x^8}{1-x^2}}$$

* Taylor's Theorem w/ Remainder:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots + \frac{f^n(a)(x-a)^n}{n!} + \underbrace{R_n(x)}_{\text{Remainder}}$$

$$R_n(x) = \underbrace{\frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!}}_{\text{All this means is THE NEXT TERM}} \text{; where "c" is between } x \,\&\, a.$$

→ when you hear "error" think, NEXT TERM!

* This can also be called the Lagrange Error Bounds.

* If $R_n(x)$, the remainder, approaches zero as $n \to \infty$, then the Taylor series converges to $f(x)$.

$$\lim_{n\to\infty} R_n(x) = 0, \text{ then } T(x) = f(x)$$

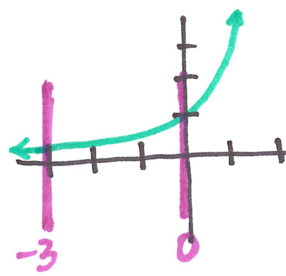
* Remainder Estimation Theorem:

$$|R_n(x)| = \frac{M\,|x-a|^{n+1}}{(n+1)!} \; ; \quad M = f^{n+1}(c) \quad \text{where "c" is a value between } x \,\&\, a.$$

Maximum Error (NEXT TERM!)

ex: #3 QR

$f(x) = 2^x \quad [-3,0]$ Find M.



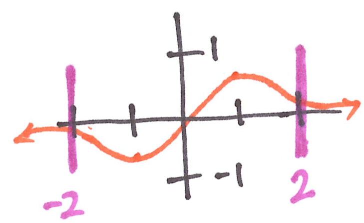
M occurs @ x=0 because it has the largest y-value

$\therefore M = f(0) = 1$

$\boxed{M=1}$

ex: #4 QR

$f(x) = \frac{x}{x^2+1} \quad [-2,2]$ Find M.



M occurs @ x=1

$f(1) = \frac{1}{2}$

$\boxed{\therefore M = \frac{1}{2}}$

ex: $\ln(1+x) \approx \underbrace{x - \frac{x^2}{2}}_{\text{2nd order poly} \;\therefore\; \text{error} \to 3^{rd} \text{ term}}$ Find the Lagrange Error when $\underbrace{|x| \le 0.1}_{-0.1 \le x \le 0.1}$ use one of these end-pts!

***Method #1:** $M \to 3^{rd}$ term

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2}$

$f'''(x) = 2(1+x)^{-3} \rightarrow 2(1+0.1)^{-3} = 1.5026...$

$2(1-0.1)^{-3} = 2.743... = M$

Larger!

$a=0$

$$\frac{f'''(c)\,|x-a|^3}{3!} = \text{error}$$

$$\frac{(2.743...)\,|-.1-0|^3}{3!} = \text{error}$$

$$\boxed{4.572 \times 10^{-4} = \text{error}}$$

*** Method #2:**

error = |actual − est| @ x = 0.1 or −0.1

$= \left|\ln(1+x) - \left(x - \frac{x^2}{2}\right)\right|$ @ x = 0.1 or −0.1

$= \left|\ln(1-.1) - \left(-.1 - \frac{(-.1)^2}{2}\right)\right|$

$$\boxed{\text{error} = 3.605 \times 10^{-4}}$$

Are these equal? No!

But... 0.0004572 -vs. 0.0003605 are VERY close!

* Do #8, 10, 20