Section 10.4: 23-53 odd

23. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n} = \Bigg(\underbrace{\frac{(x-1)^2}{4}}_{r}\Bigg)^n$

IOC: $\boxed{(-1,3)}$

$\left|\frac{(x-1)^2}{4}\right| < 1 \rightarrow |(x-1)^2| < 4 \rightarrow (x-1)^2 < 4 \rightarrow$ $x-1<2$ and $x-1>-2$

$x<3 \qquad x>-1$

$$S = \frac{a_0}{1-r} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{1}{\frac{4}{4}-\frac{(x^2-2x+1)}{4}} = \frac{1}{\frac{4-x^2+2x-1}{4}} = \boxed{\frac{4}{-x^2+2x+3}}$$

25. $\sum_{n=0}^{\infty}\Big(\underbrace{\tfrac{1}{2}\sqrt{x}-1}_{r}\Big)^n \rightarrow \left|\tfrac{1}{2}\sqrt{x}-1\right| < 1 \rightarrow$ $\tfrac{1}{2}\sqrt{x}-1<1$ and $\tfrac{1}{2}\sqrt{x}-1>-1$

$\tfrac{1}{2}\sqrt{x}<2 \qquad \tfrac{1}{2}\sqrt{x}>0$

$\sqrt{x}<4 \qquad \sqrt{x}>0$

$x<16 \qquad x>0$

IOC: $\boxed{(0,16)}$

$$S = \frac{a_0}{1-r} = \frac{1}{1-\frac{1}{2}\sqrt{x}+1} = \frac{1}{2-\frac{1}{2}\sqrt{x}}$$

$$S = \frac{1}{\frac{4}{2}-\frac{\sqrt{x}}{2}} = \frac{1}{\frac{4-\sqrt{x}}{2}} = \boxed{\frac{2}{4-\sqrt{x}}}$$

27. $\sum_{n=0}^{\infty}\Bigg(\underbrace{\frac{x^2-1}{3}}_{r}\Bigg)^n \rightarrow \left|\frac{x^2-1}{3}\right|<1 \rightarrow |x^2-1|<3 \rightarrow$ $x^2-1<3$ and $x^2-1>-3$

$x^2<4 \qquad x^2>-2$

$-2<x<2 \qquad$ Always (any x)

IOC: $\boxed{(-2,2)}$

$$S = \frac{a_0}{1-r} = \frac{1}{\frac{3}{3}-\frac{(x^2-1)}{3}}$$

$$S = \frac{1}{\frac{3-x^2+1}{3}} = \frac{1}{\frac{4-x^2}{3}} = \boxed{\frac{3}{4-x^2}}$$

29. $\sum_{n=1}^{\infty}\frac{n}{n+1} \rightarrow \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}} = 1 \neq 0 \rightarrow$ $\boxed{\text{Diverges by } n^{th} \text{ Term Test}}$

31. $\sum_{n=1}^{\infty}\frac{n^2-1}{2^n}$ Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty}\frac{(n+1)^2-1}{2^{n+1}}\cdot\frac{2^n}{n^2-1} = \lim_{n\to\infty}\frac{n^2+2n+\cancel{1}-\cancel{1}}{n^2-1}\cdot 2^{\cancel{n}-\cancel{n}-1} = 1\cdot 2^{-1} = \frac{1}{2}<1$$

$\boxed{\text{Converges by Ratio Test}}$

33. $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1} < \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \underbrace{\left(\frac{2}{3}\right)}_{r}^n$ Converges by Geometric Series with common ratio $= \frac{2}{3} < 1$.

Original series also $\boxed{\text{converges by Direct Comparison Test}}$

35. $\sum_{n=0}^{\infty} n^2 e^{-n} = \frac{n^2}{e^n}$

$\lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \cdot e^{\cancel{n}-\cancel{n}-1} = 1 \cdot e^{-1} = \frac{1}{e} < 1 \rightarrow$ $\boxed{\text{Converges by Ratio Test}}$

37. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$

$\lim_{n\to\infty} \frac{(n+4)!}{\cancel{3!}(n+1)!\,3^{n+1}} \cdot \frac{\cancel{3!}\,n!\,3^n}{(n+3)!} = \lim_{n\to\infty} \frac{n+4}{n+1} \cdot 3^{\cancel{n}-\cancel{n}-1} = 1 \cdot 3^{-1} = \frac{1}{3} < 1 \rightarrow$ $\boxed{\text{Converges by Ratio Test}}$

39. $\sum_{n=0}^{\infty} \frac{(-2)^n}{3^n} < \sum_{n=0}^{\infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$ Converges by Geometric Series with common ratio $= \frac{2}{3} < 1$.

Original series alternates signs, so the sum is smaller than $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$.

Original series also $\boxed{\text{converges by Direct Comparison Test}}$

41. $\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$

$\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3 \cdot 2^{n+1}} \cdot \frac{n^3 \cdot 2^n}{3^n} = \lim_{n\to\infty} \frac{n^3}{n^3+3n^2+3n+1} \cdot 3^{\cancel{n}+1-\cancel{n}} \cdot 2^{\cancel{n}-\cancel{n}-1}$

$1 \cdot 3^1 \cdot 2^{-1} = \frac{3}{2} > 1 \rightarrow$ $\boxed{\text{Diverges by Ratio Test}}$

43. $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

$\lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+3)} = \lim_{n\to\infty} \frac{n+1}{4n^2+10n+6}$

$\lim_{n\to\infty} \frac{\cancel{n}}{4n^{\cancel{2}}} = \lim_{n\to\infty} \frac{1}{4n} = \frac{1}{4\infty} = 0 < 1 \rightarrow$ $\boxed{\text{Converges by Ratio Test}}$

45. $\lim\limits_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$, but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

Ratio Test: $\lim\limits_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$, which is not less than 1.

47. The Ratio Test cannot determine whether the endpoints on the interval will converge or diverge. We'll learn how to check convergence at endpoints in section 10.5.

49. $\sum\limits_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$

$A(2n+1) + B(2n-1) = 6$

$n = -\frac{1}{2}: \quad -2B = 6 \to B = -3$

$n = \frac{1}{2}: \quad 2A = 6 \to A = 3$

$$\sum_{n=1}^{\infty} \left(\frac{3}{2n-1} - \frac{3}{2n+1} \right) = \underbrace{\frac{3}{1} \cancel{- \frac{3}{3}}}_{n=1} \underbrace{\cancel{+ \frac{3}{3}} \cancel{- \frac{3}{5}}}_{n=2} \underbrace{\cancel{+ \frac{3}{5}} \cancel{- \frac{3}{7}}}_{n=3} + \ldots = \frac{3}{1} = \boxed{3}$$

51. $\sum\limits_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \frac{A}{n^2} + \frac{B}{(n+1)^2}$

$A(n+1)^2 + Bn^2 = 2n + 1$

$n = -1: \quad 1B = -1 \to B = -1$

$n = 0: \quad 1A = 1 \to A = 1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) = \underbrace{\frac{1}{1^2} \cancel{- \frac{1}{2^2}}}_{n=1} \underbrace{\cancel{+ \frac{1}{2^2}} \cancel{- \frac{1}{3^2}}}_{n=2} \underbrace{\cancel{+ \frac{1}{3^2}} \cancel{- \frac{1}{4^2}}}_{n=3} = \frac{1}{1^2} = \boxed{1}$$

53. $$\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} \right) = \underbrace{\cancel{\frac{1}{\ln 3}} - \frac{1}{\ln 2}}_{n=1} \underbrace{+ \cancel{\frac{1}{\ln 4}} \cancel{- \frac{1}{\ln 3}}}_{n=2} \underbrace{+ \cancel{\frac{1}{\ln 5}} \cancel{- \frac{1}{\ln 4}}}_{n=3} + \ldots = \boxed{\frac{-1}{\ln 2}}$$