Radius of Convergence (Section 10.4)

* Convergence Theorem for Power Series: only 3 possibilities for $\sum_{n=0}^{\infty} c_n (x-a)^n$ with respect to convergence:

1) There is a positive number, R, such that $|x-a| > R$ diverges & $|x-a| < R$ converges.

   * The series may or may not converge at the end-pts: $x = a - R$ & $x = a + R$

2) If $R = \infty$, the series converges for every x.
3) If $R = 0$, the series converges @ $x = a$ & diverges elsewhere.

* Radius of Convergence (R): set of values where the series converge.

* Interval of Convergence: $|x-a| < R$ or $a - R < x < a + R$
(can be open, closed, or half-open)

Radius of Convergence

series is centered @ "a"

* n^{th} Term Test for Divergence:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist OR is different from zero.

** This test will not tell you if the series converges. If $\lim_{n\to\infty} a_n = 0$, it may or may not converge!

ex: $\sum_{n=1}^{\infty} \frac{4n^2+n^3}{7+3n^3}$ → n^{th} Term Test: $\lim_{n\to\infty} \frac{4n^2+n^3}{7+3n^3} = \frac{n^3}{3n^3} = \frac{1}{3}$

∴ the series diverges!

* <u>Direct Comparision Test</u>: Let $\sum a_n$ be a series with no negative terms.

a) $\sum a_n$ converges if $\sum c_n$ converges and $a_n \le c_n$

b) $\sum a_n$ diverges if $\sum d_n$ diverges and $a_n \ge d_n$

ex: $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ so... $\frac{x^{2n}}{(n!)^2} \le \frac{x^{2n}}{n!}$ $\qquad \frac{x^{2n}}{n!} = \frac{(x^2)^n}{n!}$ which is the Taylor series for e^{x^2}.

$\downarrow$

This converges to e^{x^2} $\therefore$ $\frac{x^{2n}}{(n!)^2}$ must also converge.

** <u>Ratio Test</u>: Let $\sum a_n$ be a series w/all positive terms.

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then a) the series converges if $L<1$
b) the series diverges if $L>1$
c) the test is inconclusive if $L=1$.

* Exp #1 on pg 512: $\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

1) Ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \frac{n}{n} = 1$

$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2} = 1$

The test is inconclusive!

2) $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \ln|x| \Big|_1^b$

$= \lim_{b\to\infty} \ln b - \cancel{\ln 1}^{\,0}$

$= \ln\infty = \infty$

$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$

$= \lim_{b\to\infty} -x^{-1} \Big|_1^b$

$= \lim_{b\to\infty} \left(\frac{-1}{b} + \frac{1}{1}\right)$

$= 0+1$

$= 1$

Exp #1 con't

3) $\boxed{\sum \frac{1}{n} \text{ diverges}}$ but $\boxed{\sum \frac{1}{n^2} \text{ converges}}$. why?

$$\sum \frac{1}{n} \geq \underbrace{\int_1^\infty \frac{1}{x}\,dx}_{\text{diverges}}$$

diverges $\therefore \sum \frac{1}{n}$ must also diverge!

$$\sum \frac{1}{n^2} \leq 1 + \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{converges}}$$

converges $\therefore \sum \frac{1}{n^2}$ must also converge

4) Both ratio tests $\Rightarrow L=1$ and one series converged & the other diverged, this is why $L=1$ is inconclusive!

*Exp #2 pg 513: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$ ← fits the terms given.

1) Ratio test: *ignore the alternating sign (just pos. terms)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}/n+1}{x^n/n} = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}$$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{\cancel{x^n}} \cdot \frac{n}{n+1} \quad (x^{n+1} = \cancel{x^n} \cdot x^1) \quad = \lim_{n\to\infty} x^1 \cdot \frac{n}{n+1} = x \lim_{n\to\infty} \frac{n}{n+1} \; (\to 1)$$

$$= x$$

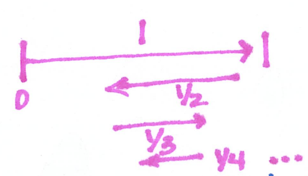
so... $|x| < 1$ to converge $\therefore R = 1$

2) Left end-pt: $x = -1$

$$= (-1) - \frac{(-1)^2}{2} + \frac{(-1)^3}{3} - \frac{(-1)^4}{4} + \dots + \frac{(-1)^{n-1}(-1)^n}{n}$$

$$= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \dots \frac{-1}{n} \to \sum_{n=1}^{\infty} -\frac{1}{n} \text{ diverges} \therefore x=-1 \text{ diverges.}$$

3) Right end-pt: $x = 1$



$$= (1) - \frac{(1)^2}{2} + \frac{(1)^3}{3} - \frac{(1)^4}{4} + \dots + \frac{(-1)^{n-1}(1)^n}{n}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n-1}}{n} \to \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges (see above)} \therefore x=1 \text{ converges}$$

The Interval of Convergence: $-1 < x \leq 1$ OR $(-1, 1]$

* Telescoping Series: a series whose partial sums eventually only have a fixed number of terms. (all others cancel out!)

ex: $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ Partial Fractions $\rightarrow$ $\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$1 = A(n+2) + B(n+1)$

let $n=-2 \rightarrow 1 = A(0) + B(-1)$

$B = -1$

let $n=-1 \rightarrow 1 = A(1) + B(0)$

$A = 1$

$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} + \frac{-1}{n+2}\right)$

$S_1 = \frac{1}{2} - \frac{1}{3}$

$S_2 = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4}$

$S_3 = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) = \frac{1}{2} - \frac{1}{5}$

$\vdots$

$S_n = \frac{1}{2} - \frac{1}{n+2}$ so... $\lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) \boxed{= \frac{1}{2}}$

* Go over geometric

10.4 Day 2