Section 10.5: QR: 1–10, Ex: 7–21 odd

Quick Review:

1. $\int_1^\infty \frac{1}{x^{4/3}}\,dx$ $\boxed{\text{converges by p-series}}$ because $\frac{4}{3} > 1$.

2. $\int_1^\infty \frac{x^2}{x^3+1}\,dx$ $\quad b_n = \frac{x^2}{x^3} = \frac{1}{x}$

Limit Comparison Test: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{x^2}{x^3+1} \cdot \frac{x}{1} = \lim\limits_{n\to\infty} \frac{x^3}{x^3+1} = 1$

$\frac{1}{x}$ diverges (by p-series w/ $p=1$), so $\frac{x^2}{x^3+1}$ also $\boxed{\text{diverges by Limit Comp. test}}$

3. $\int_1^\infty \frac{\ln x}{x}\,dx$ $\quad b_n = \frac{1}{x}$

Limit Comp. Test: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\ln x}{\cancel{x}} \cdot \frac{\cancel{x}}{1} = \lim\limits_{n\to\infty} \ln x = \ln\infty = \infty$

$\frac{1}{x}$ diverges (by p-series w/ $p=1$), so $\frac{\ln x}{x}$ also $\boxed{\text{diverges by Limit Comp. test}}$

4. $\int_1^\infty \frac{1+\overbrace{\cos x}^{\text{at most } +1}}{x^2}\,dx \le \int_1^\infty \frac{2}{x^2}\,dx$

$\frac{2}{x^2}$ converges (by p-series w/ $p=2$), so $\frac{1+\cos x}{x^2}$ also $\boxed{\text{converges by Direct Comp. test}}$

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5. $\int_1^\infty \frac{\sqrt{x}}{x+1}\,dx$ $\quad b_n = \frac{x^{1/2}}{x^1} = \frac{1}{x^{1/2}}$

Limit Comp. Test: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{x^{1/2}}{x+1} \cdot \frac{x^{1/2}}{1} = \lim\limits_{n\to\infty} \frac{x}{x+1} = 1$

$\frac{1}{x^{1/2}}$ diverges (by p-series w/ $p=1/2$), so $\frac{\sqrt{x}}{x+1}$ also $\boxed{\text{diverges by Limit Comp. test}}$

6. $f(x) = \frac{3}{x}$

$\boxed{\text{Yes}}$, positive & decreasing on $(0, \infty)$.

7. $f(x) = \frac{7x}{x^2-8}$

$\boxed{\text{Yes}}$, positive & decreasing on $(\sqrt{8}, \infty)$.

8. $f(x) = \frac{3+x^2}{3-x^2}$

$\boxed{\text{No}}$, negative & increasing as $x \to \infty$.

9. $f(x) = \frac{\sin x}{x^5}$

$\boxed{\text{No}}$, oscillates between positive & negative as $x \to \infty$.

10. $f(x) = \ln\left(\frac{1}{x}\right)$

$\boxed{\text{No}}$, negative & decreasing as $x \to \infty$.

Exercises:

7. $\sum_{n=1}^{\infty} \frac{5}{n+1}$ $\quad b_n = \frac{5}{n}$

Limit Comp. Test: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5}{n+1} \cdot \frac{n}{5} = \lim_{n\to\infty} \frac{5n}{5n+5} = 1$

$\frac{5}{n}$ diverges (by p-series w/ $p=1$), so $\frac{5}{n+1}$ also $\boxed{\text{diverges by Limit Comp. test}}$

9. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ $\quad b_n = \frac{1}{n}$

Limit Comp. Test: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n)}{\cancel{n}} \cdot \frac{\cancel{n}}{1} = \lim_{n\to\infty} \ln(n) = \ln\infty = \infty$

$\frac{1}{n}$ diverges (by p-series w/ $p=1$), so $\frac{\ln(n)}{n}$ also $\boxed{\text{diverges by Limit Comp. test}}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n} = \left(\frac{1}{\ln 2}\right)^n = (1.443)^n$

$r = 1.443 > 1$

$\boxed{\text{Diverges by Geometric Series}}$

13. $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$

$\displaystyle\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$

$\boxed{\text{Diverges by } n^{\text{th}} \text{ term test}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ $\qquad b_n = \frac{n^{1/2}}{n^2 \cancel{+1}} = n^{-3/2} = \frac{1}{n^{3/2}}$

Limit Comp. Test: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1/2}}{n^2+1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

b_n converges (by p-series w/ $p = 3/2$), so $\frac{\sqrt{n}}{n^2+1}$ also $\boxed{\text{converges by Limit Comp. test}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ $\qquad b_n = \frac{3^{n-1}}{3^n} = 3^{\cancel{n}-1-\cancel{n}} = 3^{-1} = \frac{1}{3^1}$

Limit Comp. Test: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^{n-1}+1}{3^n} \cdot \frac{3^1}{1} = \lim_{n\to\infty} \frac{3^n+3}{3^n} = 1$

$\frac{1}{3}$ diverges (bc $\displaystyle\lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$) by the n^{th} term test.

$\frac{3^{n-1}+1}{3^n}$ also $\boxed{\text{diverges by Limit Comp. test}}$

19. $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{10^n}{n^{10}}$

$\lim_{n\to\infty} \frac{10^n}{n^{10}} = \frac{10^{\infty}}{\infty^{10}} = \frac{\text{bigger}}{\text{big}} = \frac{\text{exponential}}{\text{polynomial}} = \infty \neq 0 \rightarrow \boxed{\text{Diverges by } n^{\text{th}} \text{ term test}}$

21. $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\ln(n)}{\ln(n^2)} = (-1)^{n+1} \cdot \frac{\cancel{\ln(n)}}{2\cancel{\ln(n)}} = (-1)^{n+1} \cdot \frac{1}{2}$

$\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0 \rightarrow \boxed{\text{Diverges by } n^{\text{th}} \text{ term test}}$