Section 10.5: 23-31 odd, 35-49 odd

23. $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1+n}{n^2}$

$a_n=\frac{1+n}{n^2}$, $b_n=\frac{n}{n^2}=\frac{1}{n}$ for Limit Comparison Test

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1+n}{n^2}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{n+n^2}{n^2}=1\rightarrow$ same result for a_n & b_n

b_n diverges (by p-series w/ p=1), so a_n also diverges by limit comp. test.

$\sum_{n=1}^{\infty}\frac{1+n}{n^2}$ diverges, so the series does not converge absolutely.

Check conditional convergence with Alternating Series Test:

1) Terms were positive, then alternating signs applied ✓
2) Value of terms is decreasing ✓
3) $\lim_{n\to\infty}\frac{1+n}{n^2}=\lim_{n\to\infty}\frac{n}{n^2}=\lim_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0$ ✓

The series $\boxed{\text{converges conditionally}}$ by the Alt. Series Test.

Error $\leq$ |100^{th} term| $=\frac{1+100}{100^2}=\boxed{0.0101}$

25. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\cdot\ln(n)}$

$\int_2^{\infty}\frac{1}{n\ln(n)}dn$ $\quad u=\ln(n)$, $du=\frac{1}{n}dn$, $dn=n\,du$ $\rightarrow \int\frac{1}{\cancel{n}\cdot u}\cdot\cancel{n}du=\int\frac{1}{u}du=\ln|\ln(n)|\Big|_2^{\infty}$

$\ln|\ln\infty|-\ln|\ln 2|=\ln(\infty)-\ln(0.693)=\infty+0.367=\infty\rightarrow$ Div.

$\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ diverges by the integral test, so the series does not converge absolutely.

Check conditional convergence with Alternating Series Test:

1) Terms positive, then alt. signs applied
2) Value of terms decreasing
3) $\lim_{n\to\infty}\frac{1}{n\ln(n)}=\frac{1}{\infty\cdot\ln\infty}=\frac{1}{\infty\cdot\infty}=0$ ✓

The series $\boxed{\text{converges conditionally}}$ by the Alt. Series Test.

Error $\leq$ |100^{th} term| $=\frac{1}{100\ln 100}=\boxed{0.00217}$

27. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$

Ratio Test (good for factorials & exponentials)

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot 2^{\cancel{n}-\cancel{n}-1} = \lim_{n\to\infty} (n+1) \cdot 2^{-1}$

$\lim_{n\to\infty} \frac{n+1}{2} = \frac{\infty+1}{2} = \infty > 1 \to$ series diverges by the Ratio Test,

so the series does not converge absolutely.

Check Conditional convergence with Alternating Series Test:

1) Terms positive, then alt. signs ✓

2) Value of terms decreasing × (value of terms increasing here)

The conditions of the Alt. series test have not been met, so the series does not converge conditionally. Hence, the series $\boxed{\text{diverges}}$.

29. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{1+\sqrt{n}}$

$a_n = \frac{1}{1+\sqrt{n}}$, $b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ for Limit Comparison Test

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 \to$ same result for a_n & b_n

b_n diverges (by p-series w/ $p=\frac{1}{2}$), so a_n diverges by limit comp. test.

Therefore, the series does not converge absolutely.

Check conditional convergence with Alternating Series Test:

1) Terms positive, then alt. signs applied ✓

2) Value of terms decreasing ✓

3) $\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = \frac{1}{1+\sqrt{\infty}} = \frac{1}{\infty} = 0$ ✓

The series $\boxed{\text{converges conditionally}}$ by the Alt. Series Test.

31. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = (-1)^n \cdot \frac{1}{n}$

$\cos\pi = -1,\ \cos 2\pi = +1,\ \cos 3\pi = -1,\ \cos 4\pi = +1,\ \cos 5\pi = -1$, etc.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by p-series w/ $p=1$), so the series does not converge absolutely.

Check conditional convergence with Alternating Series Test:

1) Terms positive, then alt. signs applied ✓
2) Value of terms decreasing ✓
3) $\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$ ✓

The series $\boxed{\text{converges conditionally}}$ by the Alt. Series Test.

35. $\sum_{n=0}^{\infty} x^n \rightarrow r = x \rightarrow |x| < 1 \rightarrow$ Center: $x=0$, Radius: $1 \rightarrow (-1,1)$

$x=-1$: $(-1)^n = -1+1-1+1-1+\ldots \rightarrow r=-1 \rightarrow$ Div. at $x=-1$

$x=1$: $(1)^n = 1+1+1+1+1+\ldots \rightarrow r=1 \rightarrow$ Div. at $x=1$

a) $(-1,1)$
b) $(-1,1)$
c) None

37. $\sum_{n=0}^{\infty} (-1)^n(4x+1)^n = (-1(4x+1))^n \rightarrow r = -1(4x+1) \rightarrow |\cancel{-1}(4x+1)| < 1$

$|4x+1| < 1 \rightarrow |x+1/4| < 1/4 \rightarrow$ Center: $x=-1/4$, Radius: $1/4 \rightarrow (-1/2, 0)$

$x=\frac{-1}{2}$: $(-1)^n(-2+1)^n = (-1)^n(-1)^n = (1)^n \rightarrow r=1 \rightarrow$ Div. at $x=-1/2$

$x=0$: $(-1)^n(1)^n = (-1)^n \rightarrow r=-1 \rightarrow$ Div. at $x=0$

a) $(-1/2,0)$
b) $(-1/2,0)$
c) None

39. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n} = \left(\frac{x-2}{10}\right)^n \rightarrow r = \frac{x-2}{10} \rightarrow \left|\frac{x-2}{10}\right| < 1 \rightarrow |x-2| < 10$

Cen: $x=2$, $R=10 \rightarrow (-8,12)$

$x=-8$: $\frac{(-10)^n}{10^n} = \left(\frac{-10}{10}\right)^n = (-1)^n \rightarrow r=-1 \rightarrow$ Div. at $x=-8$

$x=12$: $\frac{10^n}{10^n} = \left(\frac{10}{10}\right)^n = (1)^n \rightarrow r=1 \rightarrow$ Div. at $x=12$

a) $(-8,12)$
b) $(-8,12)$
c) None

41. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\cdot 3^n} = \frac{x^n}{n^{3/2}\cdot 3^n} \rightarrow$ Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^{3/2}\cdot 3^{n+1}} \cdot \frac{n^{3/2}\cdot 3^n}{|x|^n} = \lim_{n\to\infty} \cancel{\frac{n^{3/2}}{(n+1)^{3/2}}}^{1} \cdot |x|^{\cancel{n}+1-\cancel{n}} \cdot 3^{\cancel{n}-\cancel{n}-1} = 1\cdot|x|^1\cdot 3^{-1} = \frac{|x|}{3} = r$

$\frac{|x|}{3} < 1 \rightarrow |x| < 3 \rightarrow$ Cen: $x=0$, $R=3 \rightarrow (-3,3)$

$x=-3$: $\frac{(-3)^n}{3^n\cdot n^{3/2}} = \left(\frac{-3}{3}\right)^n \cdot \frac{1}{n^{3/2}} = (-1)^n \cdot \frac{1}{n^{3/2}} \rightarrow p = 3/2 > 1 \rightarrow$ Con. Abs. at $x=-3$

$x=3$: $\frac{\cancel{3^n}}{\cancel{3^n}\cdot n^{3/2}} = \frac{1}{n^{3/2}} \rightarrow p = 3/2 > 1 \rightarrow$ Con. Abs. at $x=3$

a) $[-3,3]$
b) $[-3,3]$
c) None

43. $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n} \rightarrow$ Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n|x+3|^n} = \lim_{n\to\infty} \cancel{\frac{n+1}{n}}^{1} \cdot |x+3|^{\cancel{n}+1-\cancel{n}} \cdot 5^{\cancel{n}-\cancel{n}-1} = 1\cdot|x+3|^1\cdot 5^{-1}$

$r = \frac{|x+3|}{5} < 1 \rightarrow |x+3| < 5 \rightarrow$ Cen: $x=-3$, $R=5 \rightarrow (-8,2)$

$x=-8$: $\frac{n(-5)^n}{5^n} = \left(\frac{-5}{5}\right)^n \cdot n = (-1)^n \cdot n = 0-1+2-3+4-5+\ldots \rightarrow$ Div. at $x=-8$

$x=2$: $n\cdot\frac{\cancel{5^n}}{\cancel{5^n}} = n = 0+1+2+3+4+\ldots \rightarrow$ Div. at $x=2$

a) $(-8,2)$
b) $(-8,2)$
c) None

45. $\sum_{n=0}^{\infty} \frac{\sqrt{n}\cdot x^n}{3^n} \rightarrow$ Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \frac{\sqrt{n+1}\cdot|x|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}\cdot|x|^n} = \lim_{n\to\infty} \cancel{\frac{\sqrt{n+1}}{\sqrt{n}}}^{1} \cdot |x|^{\cancel{n}+1-\cancel{n}} \cdot 3^{\cancel{n}-\cancel{n}-1} = 1\cdot|x|^1\cdot 3^{-1}$

$r = \frac{|x|}{3} < 1 \rightarrow |x| < 3 \rightarrow$ Cen: $x=0$, $R: 3 \rightarrow (-3,3)$

$x=-3$: $\frac{\sqrt{n}\cdot(-3)^n}{3^n} = \left(\frac{-3}{3}\right)^n\sqrt{n} = (-1)^n\sqrt{n} \rightarrow$ Div. at $x=-3$

$\lim_{n\to\infty} \sqrt{n} = \sqrt{\infty} = \infty$, so diverges by n^{th} term test (Alt. Ser. Test doesn't apply).

$x=3$: $\frac{\sqrt{n}\cdot\cancel{3^n}}{\cancel{3^n}} = \sqrt{n} \rightarrow$ Div. at $x=3$ (value of terms increasing)

a) $(-3,3)$
b) $(-3,3)$
c) None

47. $\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n \to$ Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \frac{(-2)^{n+1}(n+2)|x-1|^{n+1}}{(-2)^n (n+1)|x-1|^n} = \lim_{n\to\infty} \cancel{\frac{n+2}{n+1}}^{1} \cdot (-2)^{\cancel{n}+1-\cancel{n}} \cdot |x-1|^{\cancel{n}+1-\cancel{n}} = 1 \cdot (-2)^1 \cdot |x-1|^1$

$r = -2|x-1| \to |-2(x-1)| < 1 \to |2(x-1)| < 1 \to |x-1| < 1/2$

Cen: $x = 1$, $R = 1/2 \to (1/2, 3/2)$

$x = \frac{1}{2}$: $(-2)^n (n+1)(-\frac{1}{2})^n = (-2 \cdot -\frac{1}{2})^n (n+1) \to \cancel{(1)^n}(n+1) = n+1 \to$ Div. at $x = 1/2$ because the value of the terms is increasing.

$x = \frac{3}{2}$: $(-2)^n (n+1)(\frac{1}{2})^n = (-2 \cdot \frac{1}{2})^n (n+1) = (-1)^n (n+1) \to$ Div. at $x = 3/2$ because the value of the terms is increasing.

a) $(1/2, 3/2)$
b) $(1/2, 3/2)$
c) None

49. $\sum_{n=1}^{\infty} \frac{(x+\pi)^n}{\sqrt{n}} \to$ Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \frac{|x+\pi|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x+\pi|^n} = \lim_{n\to\infty} \cancel{\frac{\sqrt{n}}{\sqrt{n+1}}}^{1} \cdot |x+\pi|^{\cancel{n}+1-\cancel{n}} = 1 \cdot |x+\pi|^1 = |x+\pi|$

$|x+\pi| < 1 \to$ Cen: $x = -\pi$, $R = 1 \to (-\pi-1, -\pi+1)$

$x = -\pi-1$: $(\cancel{-\pi}-1+\cancel{\pi})^n \cdot \frac{1}{\sqrt{n}} = (-1)^n \cdot \frac{1}{n^{1/2}} \to p = 1/2 < 1 \to$ does not converge absolutely

Converges conditionally by Alt. Series Test at $x = -\pi-1$

$x = -\pi+1$: $(-\cancel{\pi}+1+\cancel{\pi})^n \cdot \frac{1}{\sqrt{n}} = \cancel{(1)^n} \cdot \frac{1}{n^{1/2}} = \frac{1}{n^{1/2}} \to p = 1/2 < 1 \to$ Div. at $x = -\pi+1$ by p-series

a) $[-\pi-1, -\pi+1)$
b) $(-\pi-1, -\pi+1)$
c) $x = -\pi-1$