Section 10.5: 51-65 odd

51. 13 billion = 13,000,000,000 = 13×10^9 years

Convert to number of seconds

$$\frac{13\times10^9 \cancel{yr}}{1}\cdot\frac{365 \cancel{day}}{1 \cancel{yr}}\cdot\frac{24 \cancel{hr}}{1 \cancel{day}}\cdot\frac{60 \cancel{min}}{1 \cancel{hr}}\cdot\frac{60\,s}{1 \cancel{min}} = 4.09968\times10^{17}\text{ seconds}$$

$\ln(n+1) < \text{sum} < 1+\ln(n)$

$\ln(4.09968\times10^{17}+1) = 40.555$

$1+\ln(4.09968\times10^{17}) = 41.555$

$\boxed{40.555 < \text{sum} < 41.555}$

53. (Pictures on page 517)

If the integral has an infinite area, then the series has an infinite sum and the series diverges. If the integral has a finite area, then the series has a finite sum and converges.

55. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n(x+2)^n}{3^n\cdot n!}$

Ratio Test:

$$\lim_{n\to\infty}\frac{(n+1)^{n+1}|x+2|^{n+1}}{3^{n+1}(n+1)!}\cdot\frac{3^n\cdot n!}{n^n|x+2|^n}$$

$$\lim_{n\to\infty}\frac{\cancel{n!}\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)!}\,n^n}|x+2|^{\cancel{n}+1-\cancel{n}}\cdot3^{\cancel{n}-\cancel{n}-1} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{|x+2|}{3}$$

Note: $n!(n+1) = (n+1)!$ above

$$\lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right)^n\cdot\frac{|x+2|}{3} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{|x+2|}{3} = \left(1+\overset{0}{\cancel{\frac{1}{\infty}}}\right)^{\infty}\cdot\frac{|x+2|}{3} = 1^{\infty}\cdot\frac{|x+2|}{3}$$

1^∞ is an indeterminate form, so we need to work that out.

$\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1^\infty \to$ use ln

$$n\cdot\ln\left(1+\frac{1}{n}\right) = \frac{\ln(1+\frac{1}{n})}{1/n} \to \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \frac{\ln(1+0)}{0} = \frac{0}{0} = ?$$

L'Hopital's Rule:

$$\lim_{n\to\infty}\frac{\frac{1}{1+1/n}\cdot\cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1$$

55. (continued)

$\cancel{1/n}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{1}{e} \rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 = e$

$r = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \frac{|x+2|}{3} = \frac{e|x+2|}{3} < 1 \rightarrow |x+2| < \frac{3}{e} \rightarrow \boxed{\text{Radius of Conv.} = 3/e}$

57. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by p-series w/ $p=1$, or integral test)

$\sum_{n=1}^{\infty} \frac{1}{2}\cdot\frac{1}{n}$ would grow more slowly (half the rate of the harmonic series), but half of a divergent sum is still divergent.

59. a) $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$

Limit Comparison Test with $b_n = \frac{\cancel{n}}{3n^{\cancel{2}}} = \frac{1}{3n}$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{3n^2+1}\cdot\frac{3n}{1} = \lim_{n\to\infty}\frac{3n^2}{3n^2+1} = 1 \rightarrow$ same for a_n & b_n

$b_n = \frac{1}{3n} = \frac{1}{3}\cdot\frac{1}{n} = \frac{1}{3}(\text{Div.}) = \text{Div.}$

The series A also $\boxed{\text{diverges by Limit Comp. Test}}$

b) $S = \sum_{n=1}^{\infty} \frac{n}{3n^2+1}\cdot\frac{3}{n} = \sum_{n=1}^{\infty}\frac{3n}{3n^3+n}$

Limit Comp. Test with $b_n = \frac{3n}{3n^3} = \frac{1}{n^2}$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{3n}{3n^3+n}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{3n^3}{3n^3+n} = 1 \rightarrow$ same for a_n & b_n

$b_n = \frac{1}{n^2} \rightarrow p=2>1 \rightarrow$ Converges by p-series

The series S also $\boxed{\text{converges by Limit Comp. Test}}$

61. $\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)} = \sum_{n=0}^{\infty} \frac{2^n x^n}{\ln(n+2)} = \frac{(2x)^n}{\ln(n+2)}$

Ratio Test:

$$\lim_{n\to\infty} \frac{|2x|^{n+1}}{\ln(n+3)} \cdot \frac{\ln(n+2)}{|2x|^n} = \lim_{n\to\infty} \frac{\overset{1}{\cancel{\ln(n+2)}}}{\cancel{\ln(n+3)}} \cdot |2x|^{\cancel{n}+1-\cancel{n}} = |2x| = r$$

$|2x| < 1 \to |x| < 1/2 \to$ Cen: $x = 0$, R: $1/2 \to$ IOC: $(-1/2, 1/2)$

$x = -\frac{1}{2}$: $(2 \cdot \frac{-1}{2})^n \cdot \frac{1}{\ln(n+2)} = (-1)^n \cdot \frac{1}{\ln(n+2)} \to$ Conv. Cond. by Alt. Series Test

$x = \frac{1}{2}$: $(2 \cdot \frac{1}{2})^n \cdot \frac{1}{\ln(n+2)} = \cancel{(1)^n} \cdot \frac{1}{\ln(n+2)} > \frac{1}{n} \to$ Div. by Direct Comp.

$\boxed{[-1/2, 1/2)}$

63. $\frac{1}{1+x} \to \begin{matrix} a_0 = 1 \\ r = -x \end{matrix} \to 1 - x + x^2 - x^3 + x^4 - \ldots + (-1)^n x^n$

$$\ln|1+x| = \int (1 - x + x^2 - x^3 + \ldots)\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

Given that IOC: $(-1, 1]$, so just check endpoint at $x = 1$

$x = 1$: $(-1)^{n-1} \cdot \frac{\cancel{(1)^n}}{n} = (-1)^{n-1} \cdot \frac{1}{n} \to$ $\boxed{\text{Conv. Cond. by Alt. Series Test}}$

65. $\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \ldots + \frac{1}{3^n} - \frac{1}{2^n}$

a) The value of the terms does not always decrease from one term to the next. (Condition #2)

b) $\sum_{n=1}^{\infty} \left(\frac{1}{3^n} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{3^n} - \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \to a_1 = \frac{1}{3},\ r = \frac{1}{3} \to S = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2} = \frac{1}{2}$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \to a_1 = \frac{1}{2},\ r = \frac{1}{2} \to S = \frac{1/2}{1 - 1/2} = \frac{1/2}{1/2} = 1$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} - 1 = \boxed{\frac{-1}{2}}$