Section 11.3 : 43-59 odd

43. Inside $r = 4 + 2\cos\theta$

Trace around the limacon : 0 to 360, or 0 to 2π

$$\frac{1}{2}\int_0^{2\pi}(4+2\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(16+16\cos\theta+4\cos^2\theta)\,d\theta$$

$$\int_0^{2\pi}(8+8\cos\theta+2\cos^2\theta)d\theta = \int_0^{2\pi}\left(8+8\cos\theta+2\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)\right)d\theta$$

$$\int_0^{2\pi}(8+8\cos\theta+1+\cos 2\theta)d\theta = \int_0^{2\pi}(9+8\cos\theta+\cos 2\theta)d\theta$$

$$\left(9\theta+8\sin\theta+\tfrac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = (18\pi+0+0)-(0+0+0) = \boxed{18\pi}$$

45. One petal of $r = \cos 2\theta$

Trace around one half of a petal : 0 to 45, or 0 to $\pi/4$

Double to find full petal

$$\cos^2\theta = \tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta$$

$$2\cdot\frac{1}{2}\int_0^{\pi/4}\cos^2 2\theta\, d\theta = \int_0^{\pi/4}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta\right)d\theta \qquad \cos^2 2\theta = \tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta$$

$$\left(\tfrac{1}{2}\theta+\tfrac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/4} = \left(\tfrac{\pi}{8}+0\right)-(0+0) = \boxed{\frac{\pi}{8}}$$

47. One loop of lemniscate $r^2 = 4\cos 2\theta \to r = 2\sqrt{\cos 2\theta}$

Trace around half of a loop: 0 to 45, or 0 to $\pi/4$

Double to find full loop

$$2 \cdot \frac{1}{2} \int_0^{\pi/4} (2\sqrt{\cos 2\theta})^2 \, d\theta = \int_0^{\pi/4} 4\cos 2\theta \, d\theta = 2\sin 2\theta \Big|_0^{\pi/4}$$

$$2\sin\frac{\pi}{2} - 2\sin 0 = 2(1) - 2(0) = 2 - 0 = \boxed{2}$$

49. Inside $r = 3 - 2\cos\theta$

Trace around the limacon: 0 to 360, or 0 to 2π

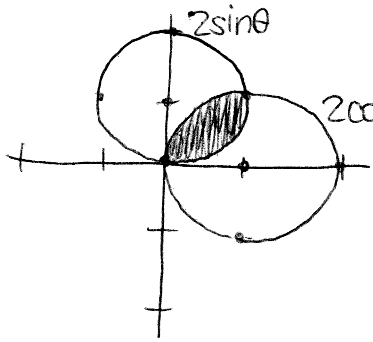
$$\frac{1}{2} \int_0^{2\pi} (3 - 2\cos\theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (9 - 12\cos\theta + 4\cos^2\theta) \, d\theta$$

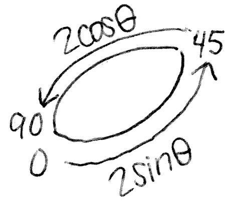
$$\int_0^{2\pi} \left(\frac{9}{2} - 6\cos\theta + 2\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\right) d\theta = \int_0^{2\pi} \left(\frac{9}{2} - 6\cos\theta + 1 + \cos 2\theta\right) d\theta$$

$$\int_0^{2\pi} \left(\frac{11}{2} - 6\cos\theta + \cos 2\theta\right) d\theta = \left(\frac{11}{2}\theta - 6\sin\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$(11\pi - 0 + 0) - (0 - 0 + 0) = \boxed{11\pi}$$

51. Shared by $r = 2\cos\theta$ and $r = 2\sin\theta$





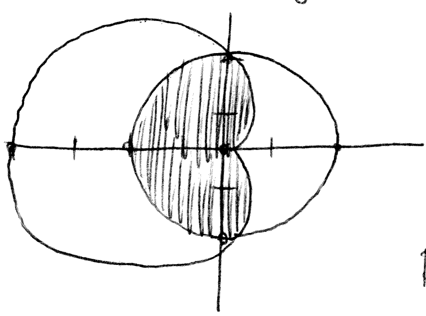
$$\frac{1}{2}\int_0^{\pi/4} 4\sin^2\theta\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} 4\cos^2\theta\, d\theta$$

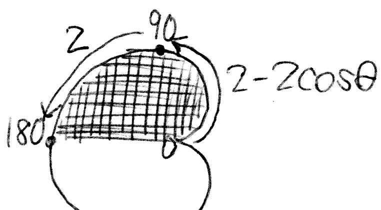
$$\int_0^{\pi/4} 2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta + \int_{\pi/4}^{\pi/2} 2\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$\int_0^{\pi/4} (1-\cos 2\theta)\,d\theta + \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\,d\theta = \left(\theta - \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/4} + \left(\theta + \tfrac{1}{2}\sin 2\theta\right)\Big|_{\pi/4}^{\pi/2}$$

$$\left(\frac{\pi}{4}-\frac{1}{2}\right) - (\cancel{0}-\cancel{0}) + \left(\frac{\pi}{2}+\cancel{0}\right) - \left(\frac{\pi}{4}+\frac{1}{2}\right) = \cancel{\frac{\pi}{4}} - \frac{1}{2} + \frac{\pi}{2} - \cancel{\frac{\pi}{4}} - \frac{1}{2} = \boxed{\frac{\pi}{2} - 1}$$

53. Shared by $r = 2$ and $r = 2 - 2\cos\theta$





Double for total area of area shared by circle & cardiod

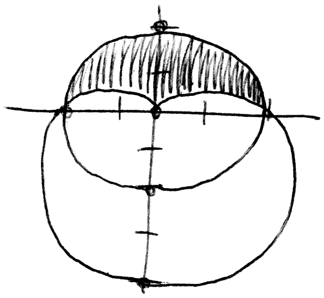
$$\cancel{2}\left[\cancel{\tfrac{1}{2}}\int_0^{\pi/2} (2-2\cos\theta)^2 d\theta + \cancel{\tfrac{1}{2}}\int_{\pi/2}^{\pi} 2^2 d\theta\right]$$

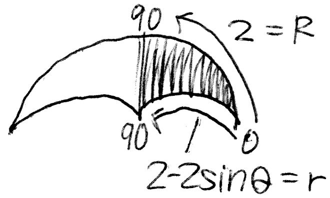
$$\int_0^{\pi/2} (4 - 8\cos\theta + 4\underbrace{\cos^2\theta}_{\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)})\,d\theta + \int_{\pi/2}^{\pi} 4\,d\theta = \int_0^{\pi/2} (4 - 8\cos\theta + 2 + 2\cos 2\theta)\,d\theta + \int_{\pi/2}^{\pi} 4\,d\theta$$

$$\left(6\theta - 8\sin\theta + \sin 2\theta\right)\Big|_0^{\pi/2} + 4\theta\Big|_{\pi/2}^{\pi}$$

$$(3\pi - 8 + \cancel{0}) - (\cancel{0} - \cancel{0} + \cancel{0}) + 4\pi - 2\pi = 3\pi - 8 + 4\pi - 2\pi = \boxed{5\pi - 8}$$

55. Inside $r=2$ and outside $r=2-2\sin\theta$





$$2\cdot\frac{1}{2}\int_0^{\pi/2}\left(2^2-(2-2\sin\theta)^2\right)d\theta$$

R^2-r^2

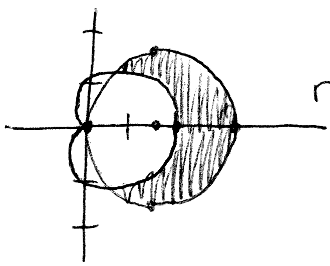
Double for total

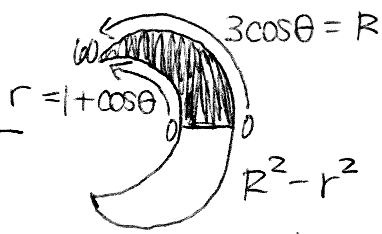
$$\cancel{2}\cdot\cancel{\frac{1}{2}}\int_0^{\pi/2}\left(\cancel{4}-\cancel{4}+8\sin\theta-4\sin^2\theta\right)d\theta$$

$$\int_0^{\pi/2}\Big(8\sin\theta-\underset{\uparrow\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)}{4\sin^2\theta}\Big)d\theta = \int_0^{\pi/2}(8\sin\theta-2+2\cos2\theta)\,d\theta = (-8\cos\theta-2\theta+\sin2\theta)\Big|_0^{\pi/2}$$

$$(\cancel{0}-\pi+\cancel{0})-(-8-\cancel{0}+\cancel{0}) = -\pi+8 = \boxed{8-\pi}$$

57. Inside $r=3\cos\theta$ and outside $r=1+\cos\theta$





R^2-r^2

Double for total

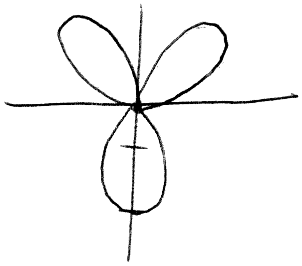
$$\cancel{2}\cdot\cancel{\frac{1}{2}}\int_0^{\pi/3}\left(9\cos^2\theta-(1+\cos\theta)^2\right)d\theta$$

$$\int_0^{\pi/3}\left(9\cos^2\theta-1-2\cos\theta-\cos^2\theta\right)d\theta = \int_0^{\pi/3}\Big(\underset{\uparrow\left(\frac{1}{2}+\frac{1}{2}\cos2\theta\right)}{8\cos^2\theta}-1-2\cos\theta\Big)d\theta$$

$$\int_0^{\pi/3}(4+4\cos2\theta-1-2\cos\theta)\,d\theta = (3\theta+2\sin2\theta-2\sin\theta)\Big|_0^{\pi/3}$$

$$\pi+\cancel{2\cdot\frac{\sqrt{3}}{2}}-\cancel{2\cdot\frac{\sqrt{3}}{2}} = \boxed{\pi}$$

59. $r = 2\sin 3\theta \rightarrow$ Rose with 3 petals



One petal: 0 to 60, or 0 to $\pi/3$; triple for total area

$$3 \cdot \frac{1}{2} \int_0^{\pi/3} 4\sin^2 3\theta$$

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$\sin^2 3\theta = \frac{1}{2} - \frac{1}{2}\cos 6\theta$

$$\frac{3}{2}\int_0^{\pi/3} 4\left(\frac{1}{2} - \frac{1}{2}\cos 6\theta\right) d\theta = \frac{3}{2}\int_0^{\pi/3} (2 - 2\cos 6\theta)\, d\theta = \int_0^{\pi/3} (3 - 3\cos 6\theta)\, d\theta$$

$$\left(3\theta - \frac{1}{2}\sin 6\theta\right)\Big|_0^{\pi/3} = (\pi - \cancel{0}) - (\cancel{0} - \cancel{0}) = \boxed{\pi}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$y = r\sin\theta = 2\sin 3\theta \sin\theta$

$x = r\cos\theta = 2\sin 3\theta \cos\theta$

$$\frac{dy}{dx} = \frac{2\sin 3\theta \cos\theta + \sin\theta \cdot 6\cos 3\theta}{-2\sin 3\theta \sin\theta + \cos\theta \cdot 6\cos 3\theta} \quad \text{at } \theta = \pi/4$$

$$\frac{dy}{dx} = \frac{\cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot 6 \cdot -\frac{\sqrt{2}}{2}}{-\cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot 6 \cdot -\frac{\sqrt{2}}{2}} = \frac{1 - 3}{-1 - 3} = \frac{-2}{-4} = \boxed{\frac{1}{2}}$$