Calculus of Polar Curves (Section 11.3)

* Polar to Rectangular Conversions: $x = r\cos\theta$ $\quad x^2+y^2=r^2$
$y = r\sin\theta$ $\quad \theta = \tan^{-1}(y/x)$

* Slopes of Polar Curves: $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

ex: $r = \cos 2\theta$ @ $\theta = \pi/4$

$x = r\cdot\cos\theta$ $\qquad y = r\sin\theta$

$x = \cos 2\theta \cdot \cos\theta$ $\qquad y = \cos 2\theta \cdot \sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(\cos 2\theta)\cos\theta + \sin\theta(-2\sin 2\theta)}{(\cos 2\theta)(-\sin\theta) + \cos\theta(-2\sin 2\theta)}$$

@ $\theta = \pi/4 \rightarrow$

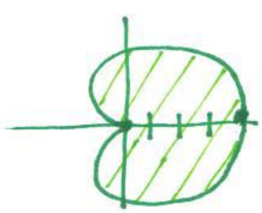
$$\frac{dy}{dx} = \frac{(\cancelto{0}{\cos \pi/2})(\cos \pi/4) + (\sin \pi/4)(-2\sin \pi/2)}{(\cancelto{0}{\cos \pi/2})(-\sin \pi/4) + (\cos \pi/4)(-2\sin \pi/2)}$$

$$\frac{dy}{dx} = \frac{0 - 2(\sqrt{2}/2)(1)}{0 - 2(\sqrt{2}/2)(1)}$$

$$\boxed{\frac{dy}{dx} = 1}$$

* Area in Polar Coordinates: $A = \displaystyle\int_\alpha^\beta \frac{1}{2} r^2\, d\theta = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\, d\theta$

ex: $r = 2 + 2\cos\theta$ Find the area inside the cardiod.



$$\int_0^{2\pi} \frac{1}{2}(2+2\cos\theta)^2\, d\theta \quad \text{OR} \quad 2\int_0^{\pi} \frac{1}{2}(2+2\cos\theta)^2\, d\theta$$

$$= \int_0^{\pi} (4 + 8\cos\theta + 4\cos^2\theta)\, d\theta$$

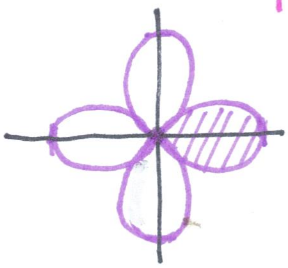
$$= 4\theta + 8\sin\theta + \frac{\cancelto{2}{4}\theta}{\cancel{2}} + \frac{\cancel{4}\sin 2\theta}{\cancel{4}} \Big|_0^{\pi}$$

$$= [4\pi + 0 + 2\pi + 0] - (0)$$

$$\boxed{= 6\pi}$$

ex: $r = \cos 2\theta$ Find the area of one petal.

petals are on axis'
Total of 4 petals.



* you could spend time figuring out that one petal goes from $-\pi/4$ to $\pi/4$ or use symmetry & double the integral from 0 to $\pi/4$.

OR $\frac{1}{4}\int_0^{2\pi} \frac{1}{2}(\cos 2\theta)^2 d\theta$

$\frac{1}{8}\int_0^{2\pi} \cos^2 2\theta \, d\theta$ ← $\int$ Table #59

$= \frac{1}{8}\left(\frac{\theta}{2} + \frac{\sin 2(2\theta)}{4(2)}\right)\Big|_0^{2\pi}$

$= \frac{1}{16}\left[2\pi + \cancel{\frac{\sin 8\pi}{4}}^{\,0} - 0\right]$

$= 2\pi/16 = \boxed{\pi/8}$

*Area between Two Polar Curves:

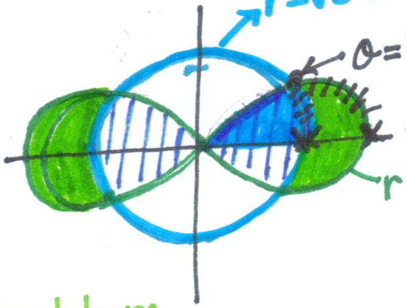
$$A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\,d\theta$$

*** FOLLOW THE PATH OF THE CURVES!!!

ex: $r^2 = 6\cos 2\theta$ → Leminiscate

$r = \sqrt{3}$

Find the indicated areas.



$\frac{3}{6} = \frac{6\cos 2\theta}{6}$

$1/2 = \cos 2\theta$

↓

$1/2 = \cos\theta$

$\theta = \pi/3$

So $1/2 = \cos 2\theta$

$\theta = \pi/6$

Solid Green Area

* We will subtract here because we are traveling on BOTH curves @ the same time! Use right - left.

$A = 4\int_0^{\pi/6} \frac{1}{2}(L^2 - C^2)\,d\theta$

$A = 2\int_0^{\pi/6} (6\cos 2\theta - 3)\,d\theta$

$= 6\sin 2\theta - 6\theta \Big|_0^{\pi/6}$

$= 6\sin \pi/3 - \pi - 0$

$\boxed{= 3\sqrt{3} - \pi}$

Striped Blue Lines → Now we are only traveling on the Leminiscate so we don't subtract!

$A = 4\left[\int_0^{\pi/6} \frac{1}{2} C^2 d\theta + \int_{\pi/6}^{\pi/4} \frac{1}{2} L^2 d\theta\right]$

$A = 2\left[\int_0^{\pi/6} (3)\,d\theta + \int_{\pi/6}^{\pi/4} 6\cos 2\theta \, d\theta\right]$

$= 6\theta \Big|_0^{\pi/6} + 6\sin 2\theta \Big|_{\pi/6}^{\pi/4}$

$= \pi - 0 + 6\sin \pi/2 - 6\sin \pi/3$

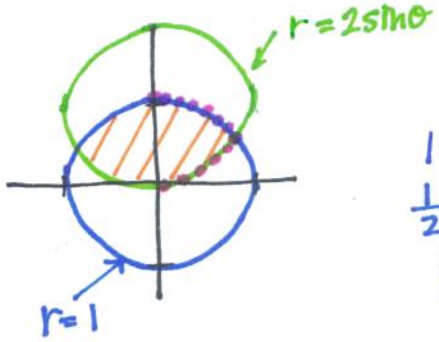
$= \pi + 6 - 6(\sqrt{3}/2)$

$\boxed{= \pi + 6 - 3\sqrt{3}}$

ex: $r=1$

$r = 2\sin\theta$

Find the area shared by the circles.



$1 = 2\sin\theta$

$\frac{1}{2} = \sin\theta$

$\theta = \pi/6$ (intersection pt.)

* You will add 2 integrals together because you are traveling on one curve & then the other. (NOT BOTH @ the same time!)

$$A = \cancel{2}\left[\int_0^{\pi/6} \cancel{\tfrac{1}{2}}(2\sin\theta)^2 d\theta + \int_{\pi/6}^{\pi/2} \cancel{\tfrac{1}{2}}(1)^2 d\theta\right]$$

$$A = \int_0^{\pi/6} 4\sin^2\theta \, d\theta + \int_{\pi/6}^{\pi/2} 1 \, d\theta$$

$$= 4\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]\Big|_0^{\pi/6} + \theta \Big|_{\pi/6}^{\pi/2}$$

$$= 4\left[\left(\frac{\pi}{12} - \frac{\sin \pi/3}{4}\right) - 0\right] + \left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

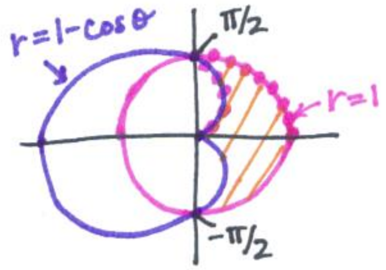
$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{3} \quad \boxed{= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

ex: $r=1$

$r = 1-\cos\theta$

$-\pi/2 \le \theta \le \pi/2$

Find the area inside the circle & outside the cardiod.



* You will subtract here because you travel on BOTH curves at the same time!

* use "Right - Left"

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[(1)^2 - (1-\cos\theta)^2\right] d\theta$$

$$= \cancel{2}\int_0^{\pi/2} \cancel{1/2}\left[\cancel{1} - \cancel{1} + 2\cos\theta - \cos^2\theta\right] d\theta = \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= 2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{\pi/2}$$

$$= \left(2\sin\pi/2 - \frac{\pi}{4} - \frac{\sin\pi}{4}\right) - (0) \quad \boxed{= 2 - \pi/4}$$

#31 $r = \sec\theta \cdot \tan\theta$

$$r = \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta} \rightarrow \text{Parabola}$$

#43 $r = 4 + 2\cos\theta$

$$A = \int_0^{2\pi} \frac{1}{2}(4+2\cos\theta)^2 d\theta = \cancel{2}\int_0^{\pi} \cancel{\tfrac{1}{2}}(4+2\cos\theta)^2 d\theta$$

$$= \int_0^{\pi} (16 + 16\cos\theta + 4\cos^2\theta)\, d\theta$$

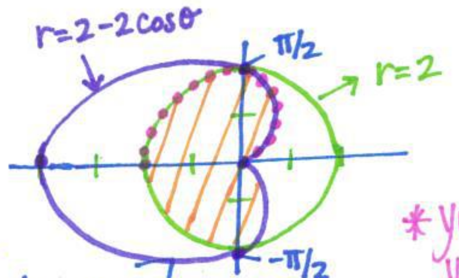
$$= 16\theta + 16\sin\theta + \frac{4\theta}{2} + \frac{\cancel{4}\sin(2\theta)}{\cancel{4}} \Big|_0^{\pi}$$

$$= (16\pi + 0 + 2\pi + 0) - (0) \boxed{= 18\pi}$$

#53 $r = 2$

$r = 2 - 2\cos\theta$

Find the area <u>shared</u>.



* You add these because you travel on one curve & then the other.

$$A = \cancel{2}\left[\int_0^{\pi/2} \cancel{\tfrac{1}{2}}(2-2\cos\theta)^2 d\theta + \int_{\pi/2}^{\pi} \cancel{\tfrac{1}{2}}(2)^2 d\theta\right]$$

$$= \int_0^{\pi/2} 4 - 8\cos\theta + 4\cos^2\theta + \int_{\pi/2}^{\pi} 4\, d\theta$$

$$= 4\theta - 8\sin\theta + \frac{4\theta}{2} + \frac{\cancel{4}\sin 2\theta}{\cancel{4}} \Big|_0^{\pi/2} + 4\theta \Big|_{\pi/2}^{\pi}$$

$$= \cancel{2\pi} - 8\sin\tfrac{\pi}{2} + \pi + 0 - (0) + 4\pi - \cancel{2\pi}$$

$$\boxed{= 5\pi - 8}$$