Section 4.2 Key - Day 1: 1-57 e.o.o.

1. $x^2y + xy^2 = 6$

$x^2\frac{dy}{dx} + y\cdot 2x + x\cdot 2y\frac{dy}{dx} + y^2 = 0$

$\frac{dy}{dx}(x^2+2xy) = -2xy - y^2$

$\frac{dy}{dx} = \boxed{\frac{-2xy-y^2}{x^2+2xy}}$

5. $x = \tan y$

$1 = \sec^2 y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \boxed{\cos^2 y}$

9. $x^2+y^2 = 13$ at $(-2,3)$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = \boxed{\frac{-x}{y}} = \frac{-(-2)}{3} = \boxed{\frac{2}{3}}$

13. $x^2y - xy^2 = 4$

$x^2\frac{dy}{dx} + y\cdot 2x - x\cdot 2y\frac{dy}{dx} - y^2 = 0$

$\frac{dy}{dx}(x^2 - 2xy) = y^2 - 2xy$

$\frac{dy}{dx} = \boxed{\frac{y^2-2xy}{x^2-2xy}}$

$x^2 - 2xy \neq 0$

$x(x-2y) \neq 0$

$\boxed{x \neq 0}$ $\quad x - 2y \neq 0$

$2y \neq x$

$\boxed{y \neq \frac{x}{2}}$

17. $x^2 + xy - y^2 = 1 \quad (2,3)$

$2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x-2y) = -2x - y$

$\frac{dy}{dx} = \frac{-2x-y}{x-2y}$ at $(2,3) = \frac{-2(2)-3}{2-2(3)} = \frac{-4-3}{2-6} = \frac{-7}{-4} = \frac{7}{4}$

Tan: $y - 3 = \frac{7}{4}(x-2)$

$y - 3 = \frac{7}{4}x - \frac{7}{2}$

$\boxed{y = \frac{7}{4}x - \frac{1}{2}}$

Normal: $y - 3 = \frac{-4}{7}(x-2)$

$y - 3 = -\frac{4}{7}x + \frac{8}{7}$

$\boxed{y = -\frac{4}{7}x + \frac{29}{7}}$

21. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at $(-1,0)$

$$12x + 3x\frac{dy}{dx} + 3y + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x+4y+17) = -12x-3y$$

$$\frac{dy}{dx} = \frac{-12x-3y}{3x+4y+17} \text{ at } (-1,0) = \frac{12-0}{-3+0+17} = \frac{12}{14} = \frac{6}{7}$$

Tan: $y - 0 = \frac{6}{7}(x+1)$ $\boxed{y = \frac{6}{7}x + \frac{6}{7}}$

Normal: $y - 0 = \frac{-7}{6}(x+1)$ $\boxed{y = \frac{-7}{6}x - \frac{7}{6}}$

25. $y = 2\sin(\pi x - y)$ at $(1,0)$

$$\frac{dy}{dx} = 2\cos(\pi x - y)\left(\pi - \frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = 2\pi\cos(\pi x - y) - 2\cos(\pi x - y)\frac{dy}{dx}$$

$$1\frac{dy}{dx} + 2\cos(\pi x - y)\frac{dy}{dx} = 2\pi\cos(\pi x - y)$$

$$\frac{dy}{dx} = \frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)} \text{ at } (1,0) = \frac{2\pi\cos\pi}{1+2\cos\pi} = \frac{2\pi(-1)}{1+2(-1)} = \frac{-2\pi}{-1} = 2\pi$$

Tan: $y - 0 = 2\pi(x-1)$ $\boxed{y = 2\pi x - 2\pi}$

Normal: $y - 0 = \frac{-1}{2\pi}(x-1)$ $\boxed{y = \frac{-1}{2\pi}x + \frac{1}{2\pi}}$

29. $y^2 = x^2 + 2x$

$$2y\frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx} = \frac{2x+2}{2y} = \frac{\cancel{2}(x+1)}{\cancel{2}y} = \boxed{\frac{x+1}{y}}$$

Replace $\frac{dy}{dx}$ with the answer from above

$$\frac{d^2y}{dx^2} = \frac{y(1) - (x+1)\frac{dy}{dx}}{y^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2} = \frac{\frac{y \cdot y}{1 \cdot y} - \frac{(x+1)^2}{y}}{y^2} = \frac{\frac{y^2 - (x+1)^2}{y}}{\frac{y^2}{1}}$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - (x+1)^2}{y} \cdot \frac{1}{y^2} = \frac{y^2 - (x+1)^2}{y^3} = \frac{\cancel{x^2} + \cancel{2x} - \cancel{x^2} - \cancel{2x} - 1}{y^3} = \boxed{\frac{-1}{y^3}}$$

Replace y^2 w/ original equation

33. $y = \sqrt[3]{x} = x^{1/3}$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \boxed{\frac{1}{3x^{2/3}}}$$

37. $y = x\sqrt{x^2+1} = x(x^2+1)^{1/2}$

$$\frac{dy}{dx} = x \cdot \cancel{\tfrac{1}{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x + (x^2+1)^{1/2} \cdot 1 = \boxed{\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}}$$

41. $y = 3(\csc x)^{3/2}$

$$\frac{dy}{dx} = \frac{9}{2}(\csc x)^{1/2}(-\csc' x \cot x) = \boxed{-\frac{9}{2}\csc^{3/2} x \cot x}$$

45. $y^4 = y^2 - x^2$

$$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$

$$\frac{dy}{dx}(4y^3 - 2y) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y^3-2y} = \frac{-x}{2y^3-y}$$

At $(\sqrt{3}/4, \sqrt{3}/2)$: $\dfrac{-\sqrt{3}/4}{2\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{\sqrt{3}}{2}} = \dfrac{-\sqrt{3}/4}{\frac{3\sqrt{3}}{4} - \frac{2\sqrt{3}}{4}} = \dfrac{-\sqrt{3}/4}{\sqrt{3}/4} = \boxed{-1}$

At $(\sqrt{3}/4, 1/2)$: $\dfrac{-\sqrt{3}/4}{2\left(\frac{1}{2}\right)^3 - \frac{1}{2}} = \dfrac{-\sqrt{3}/4}{\frac{1}{4} - \frac{1}{2}} = \dfrac{-\sqrt{3}/4}{-1/4} = -\dfrac{\sqrt{3}}{\cancel{4}} \cdot \dfrac{-\cancel{4}}{1} = \boxed{\sqrt{3}}$

49. $x^2 + xy + y^2 = 7 \rightarrow y = 0$

$x^2 + 0 + 0 = 7$

$x = \pm\sqrt{7} \rightarrow \boxed{(\sqrt{7}, 0) \text{ and } (-\sqrt{7}, 0)}$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y) = -2x - y$$

$$\frac{dx}{dx} = \frac{-2x-y}{x+2y}$$

At $(\sqrt{7}, 0)$: $\dfrac{-2\sqrt{7} - \cancel{0}}{\sqrt{7} + \cancel{0}} = \dfrac{-2\cancel{\sqrt{7}}}{\cancel{\sqrt{7}}} = \boxed{-2}$

At $(-\sqrt{7}, 0)$: $\dfrac{2\sqrt{7} - 0}{-\sqrt{7} + 0} = \dfrac{2\cancel{\sqrt{7}}}{-\cancel{\sqrt{7}}} = \boxed{-2}$

53. $v = 8\sqrt{s-t} + 1$

$v = 8(s-t)^{1/2} + 1$

$$a = 4(s-t)^{-1/2}(\underset{\substack{\uparrow \\ v}}{s'} - 1) = \frac{4}{\cancel{\sqrt{s-t}}} \cdot 8\cancel{\sqrt{s-t}} + \cancel{1} - \cancel{1} = \boxed{32\text{ ft/s}^2}$$

57. $xy + 2x - y = 0$

Parallel to $2x+y=0 \rightarrow y=-2x \rightarrow m=-2$ parallel to normal lines $\rightarrow$ Tangent slope $= \frac{1}{2}$

$x\frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0$

$\frac{dy}{dx}(x-1) = -y-2$

$\frac{dy}{dx} = \frac{-y-2}{x-1} \rightarrow \frac{-y-2}{x-1} \times \frac{1}{2} \rightarrow$ $-2y-4 = x-1$, $x=-2y-3$

$xy+2x-y=0$ with $x=-2y-3$

$(-2y-3)y + 2(-2y-3) - y = 0$

$-2y^2-3y-4y-6-y=0$

$-2y^2-8y-6=0$

$2y^2+8y+6=0$

$y^2+4y+3=0$

$(y+3)(y+1)=0$

$y=-3,\ y=-1$

$x=-2(-3)-3 \quad x=-2(-1)-3$

$x=3 \quad x=-1$

$(3,-3) \quad (-1,-1)$

$m=-2 \quad m=-2$

$y+3=-2(x-3) \quad y+1=-2(x+1)$

$y+3=-2x+6 \quad y+1=-2x-2$

$\boxed{y=-2x+3} \quad \boxed{y=-2x-3}$