Section 4.2 Key - Day 2 : 3-55 e.o.o., 65

3. $y^2 = \dfrac{x-1}{x+1}$

$$2y\frac{dy}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2} = \frac{\cancel{x}+1-\cancel{x}+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{2}}{(x+1)^2\cdot\cancel{2}y} = \boxed{\frac{1}{y(x+1)^2}}$$

7. $x+\tan(xy) = 0$

$$1+\sec^2(xy)\left(x\frac{dy}{dx}+y\right) = 0$$

$$1+x\frac{dy}{dx}\sec^2(xy)+y\sec^2(xy) = 0$$

$$\frac{dy}{dx} = \frac{-1-y\sec^2(xy)}{x\sec^2(xy)} = \frac{-1}{x\sec^2(xy)} - \frac{y\cancel{\sec^2(xy)}}{x\cancel{\sec^2(xy)}} = \boxed{\frac{-\cos^2(xy)}{x} - \frac{y}{x}}$$

11. $(x-1)^2+(y-1)^2 = 13$ at $(3,4)$

$$2(x-1)+2(y-1)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}(x-1)}{\cancel{2}(y-1)} = \boxed{\frac{-x+1}{y-1}} \text{ at } (3,4) = \frac{-3+1}{4-1} = \boxed{\frac{-2}{3}}$$

15. $x^3+y^3 = xy$

$$3x^2+3y^2\frac{dy}{dx} = x\frac{dy}{dx}+y$$

$$\frac{dy}{dx}(3y^2-x) = y-3x^2$$

$$\frac{dy}{dx} = \boxed{\frac{y-3x^2}{3y^2-x}}$$

$3y^2-x \neq 0$

$\boxed{3y^2 \neq x}$

19. $x^2y^2 = 9$ at $(-1,3)$

$$x^2\cdot 2y\frac{dy}{dx}+y^2\cdot 2x = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}\cancel{x}y^{\cancel{2}}}{\cancel{2}x^{\cancel{2}}\cancel{y}} = \frac{-y}{x} \text{ at } (-1,3) = \frac{-3}{-1} = 3 = \text{Tan} \to \text{Normal} = -1/3$$

Tan: $y-3 = +3(x+1)$

$y-3 = +3x+3$

$\boxed{y = +3x+6}$

Norm: $y-3 = -1/3\,(x+1)$

$y-3 = -\frac{1}{3}x - \frac{1}{3}$

$\boxed{y = -\frac{1}{3}x+\frac{8}{3}}$

23. $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$

$$2x\frac{dy}{dx} + 2y + \pi\cos y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2y}{2x+\pi\cos y} \text{ at } (1,\pi/2) = \frac{-2(\pi/2)}{2(1)+\pi\cos\pi/2} = \frac{-\pi}{2+0} = \frac{-\pi}{2} \text{ Tan} \rightarrow \text{Normal} = \frac{2}{\pi}$$

Tan: $y - \pi/2 = -\pi/2(x-1)$

$$y - \pi/2 = -\frac{\pi}{2}x + \frac{\pi}{2}$$

$$\boxed{y = \frac{-\pi}{2}x + \pi}$$

Normal: $y - \pi/2 = \frac{2}{\pi}(x-1)$

$$y - \pi/2 = \frac{2}{\pi}x - \frac{2}{\pi}$$

$$\boxed{y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}}$$

27. $x^2+y^2=1$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = \boxed{\frac{-x}{y}}$$

$$\frac{d^2y}{dx^2} = \frac{y(-1)-(-x)\frac{dy}{dx}}{y^2} = \frac{-\frac{y}{1}+x\left(\frac{-x}{y}\right)}{y^2} = \frac{\frac{-y^2}{y}-\frac{x^2}{y}}{y^2} = \frac{\frac{-y^2-x^2}{y}}{\frac{y^2}{1}}$$

$$\frac{d^2y}{dx^2} = \frac{-y^2-x^2}{y}\cdot\frac{1}{y^2} = \frac{-y^2-x^2}{y^3} = \frac{-1(x^2+y^2)}{y^3} = \frac{-1(1)}{y^3} = \boxed{\frac{-1}{y^3}}$$

31. $y = x^{9/4}$

$$\frac{dy}{dx} = \boxed{\frac{9}{4}x^{5/4}}$$

35. $y = (2x+5)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{\cancel{2}}(2x+5)^{-3/2}\cdot\cancel{2} = \boxed{\frac{-1}{(2x+5)^{3/2}}}$$

39. $y = \sqrt{1-\sqrt{x}} = (1-x^{1/2})^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(1-x^{1/2})^{-1/2}\cdot -\frac{1}{2}x^{-1/2} = \boxed{\frac{-1}{4\sqrt{x}\cdot\sqrt{1-\sqrt{x}}}}$$

43. a) $f(x) = \frac{3}{2}x^{2/3} - 3$

$$f'(x) = x^{-1/3}$$

$$f''(x) = -\frac{1}{3}x^{-4/3} \rightarrow \text{No}$$

b) $f(x) = \frac{9}{10}x^{5/3} - 7$

$$f'(x) = \frac{3}{2}x^{2/3}$$

$$f''(x) = x^{-1/3} \rightarrow \boxed{\text{Yes}}$$

c) $f''(x) = x^{-1/3}$

$$f'''(x) = -1/3\, x^{-4/3} \rightarrow \boxed{\text{Yes}}$$

d) $f'(x) = \frac{3}{2}x^{2/3} + 6$

$$f''(x) = x^{-1/3} \rightarrow \boxed{\text{Yes}}$$

47. $x^3y^2 = \cos(\pi y)$ at $(-1,1)$

a) $(-1)^3 1^2 = \cos(\pi)$

$-1 = \cos\pi$ ✓

b) $x^3 \cdot 2y\frac{dy}{dx} + y^2 \cdot 3x^2 = -\sin(\pi y)\cdot \pi\frac{dy}{dx}$

$\frac{dy}{dx}\left(2x^3y + \pi\sin(\pi y)\right) = -3x^2y^2$

$\frac{dy}{dx} = \frac{-3x^2y^2}{2x^3y+\pi\sin(\pi y)}$ at $(-1,1) = \frac{-3\cdot1\cdot1}{2\cdot-1\cdot1+\cancel{\pi\cdot 0}} = \frac{-3}{-2} = \boxed{\frac{3}{2}}$

51. $2x^2+3y^2=5$

$4x+6y\frac{dy}{dx}=0$

$\frac{dy}{dx} = \frac{-4x}{6y} = \frac{-2x}{3y}$ at $(1,1) = \frac{-2}{3}$; at $(1,-1) = \frac{-2}{-3} = \frac{2}{3}$

At $(1,1)$: $-2/3 \perp 3/2$ ✓

At $(1,-1)$: $2/3 \perp -3/2$ ✓

$y^2 = x^3$

$2y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2y}$ at $(1,1) = \frac{3}{2}$; at $(1,-1) = \frac{3}{-2}$

55. $x^3+y^3-9xy=0$

a) $3x^2+3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9y = 0$

$\frac{dy}{dx}(3y^2-9x) = 9y-3x^2$

$\frac{dy}{dx} = \frac{9y-3x^2}{3y^2-9x} = \frac{3y-x^2}{y^2-3x}$ at $(4,2) = \frac{6-16}{4-12} = \frac{-10}{-8} = \boxed{\frac{5}{4}}$; at $(2,4) = \frac{12-4}{16-6} = \frac{8}{10} = \boxed{\frac{4}{5}}$

b) $\frac{dy}{dx}=0$ when numerator $=0$ → $3y-x^2=0$ → $x^2=3y$ → $y=\frac{1}{3}x^2$

$x^3+\left(\frac{1}{3}x^2\right)^3 - 9x\left(\frac{1}{3}x^2\right) = 0$

$x^3+\frac{1}{27}x^6 - 3x^3 = 0$

$\frac{1}{27}x^6 - 2x^3 = 0$

$x^3\left(\frac{1}{27}x^3-2\right)=0$

$x^3 = 0$ $\qquad \frac{1}{27}x^3 - 2 = 0$

$x = 0$ $\qquad \frac{1}{27}x^3 = 2$

$x^3 = 54$

$x = \sqrt[3]{54}$

$x = 3\sqrt[3]{2}$

$y = \frac{1}{3}x^2 = \frac{1}{3}(3\sqrt[3]{2})^2 = \frac{1}{3}\cdot 9\cdot 2^{2/3} = 3\cdot\sqrt[3]{4}$

$\boxed{(3\sqrt[3]{2}, 3\sqrt[3]{4})}$

55. c) Veritcal tangent when denominator = 0 → $y^2 - 3x = 0$

$$3x = y^2 \rightarrow x = \tfrac{1}{3}y^2$$
$$\left(\tfrac{1}{3}y^2\right)^3 + y^3 - 9\left(\tfrac{1}{3}y^2\right)y = 0$$
$$\tfrac{1}{27}y^6 + y^3 - 3y^3 = 0$$
$$\tfrac{1}{27}y^6 - 2y^3 = 0$$
$$y^3\left(\tfrac{1}{27}y^3 - 2\right) = 0$$
$$y^3 = 0$$
$$y = 0$$

$$\tfrac{1}{27}y^3 - 2 = 0$$
$$\tfrac{1}{27}y^3 = 2$$
$$y^3 = 54$$
$$y = \sqrt[3]{54}$$
$$y = 3\sqrt[3]{2}$$

$$x = \tfrac{1}{3}\left(3\sqrt[3]{2}\right)^2$$
$$x = \tfrac{1}{3}\cdot 9 \cdot \sqrt[3]{4}$$
$$x = 3\sqrt[3]{4}$$
$$\boxed{\left(3\sqrt[3]{4}, 3\sqrt[3]{2}\right)}$$

65. a) $\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1$

$$\frac{2}{a^2}x + \frac{2}{b^2}y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2/a^2\, x}{2/b^2\, y} = \frac{-\cancel{2}x}{a^2}\cdot\frac{b^2}{\cancel{2}y} = \frac{-b^2x}{a^2y} \text{ at } (x_1, y_1) = \frac{-b^2x_1}{a^2y_1} \text{ slope}$$
$$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$$
$$a^2y_1\left(\frac{y}{1} - \frac{y_1}{1}\right) = \left(\frac{-b^2x_1x}{a^2y_1} + \frac{b^2x_1^2}{a^2y_1}\right)a^2y'$$
$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$$
$$\frac{a^2y_1y + b^2x_1x}{a^2b^2} = \frac{b^2x_1^2 + a^2y_1^2}{a^2b^2}$$
$$\frac{y_1y}{b^2} + \frac{x_1x}{a^2} = \underbrace{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}_{\text{Original equation at } (x_1, y_1)}$$
$$\boxed{\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1}$$

b) Same as part A, except subtraction instead of addition.