Section 5.6 - Part 1: 9-22 All

9. $\frac{dl}{dt} = -2$ cm/s, $\frac{dw}{dt} = 2$ cm/s, $l = 12$ cm, $w = 5$ cm

a) $A = lw$

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt} = 12\cdot 2 + 5(-2) = 24 - 10 = \boxed{14 \text{ cm}^2/\text{s}}$$

b) $P = 2l + 2w$

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt} = 2\cdot(-2) + 2(2) = -4 + 4 = \boxed{0 \text{ cm/s}}$$

c) $d^2 = l^2 + w^2$

$$2d\frac{dd}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt}$$

$$d\frac{dd}{dt} = l\frac{dl}{dt} + w\frac{dw}{dt} \rightarrow \frac{dd}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{d} = \frac{12(-2) + 5(2)}{13} = \boxed{\frac{-14}{13} \text{ cm/s}}$$

d) The length of the diagonal is decreasing bc the rate of change is negative. The area is increasing bc the rate of change is positive. The perimeter is neither increasing nor decreasing bc the rate of change is 0.

10. $\frac{dx}{dt} = 1$ m/s, $\frac{dy}{dt} = -2$ m/s, $\frac{dz}{dt} = 1$ m/s, $x = 4$ m, $y = 3$ m, $z = 2$ m

a) $V = xyz = (xy)z$

$$\frac{dV}{dt} = xy\frac{dz}{dt} + z\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = 4\cdot 3\cdot 1 + 2(4(-2) + 3\cdot 1) = 12 + 2(-5) = \boxed{2 \text{ m}^3/\text{s}}$$

b) $S = 2xy + 2xz + 2yz$

$$\frac{dS}{dt} = 2x\frac{dy}{dt} + y\cdot 2\frac{dx}{dt} + 2x\frac{dz}{dt} + z\cdot 2\frac{dx}{dt} + 2y\frac{dz}{dt} + z\cdot 2\frac{dy}{dt}$$

$$\frac{dS}{dt} = 2(4(-2) + 3(1) + 4(1) + 2(1) + 3(1) + 2(-2)) = 2(-8 + 3 \cancel{+4} + 2 + 3 \cancel{-4}) = 2\cdot 0 = \boxed{0 \text{ m}^2/\text{s}}$$

c) $s = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$

$$\frac{ds}{dt} = \frac{\cancel{1}}{\cancel{2}}(x^2 + y^2 + z^2)^{-1/2}\left(\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} + \cancel{2}z\frac{dz}{dt}\right)$$

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}} = \frac{4(1) + 3(-2) + 2(1)}{\sqrt{4^2 + 3^2 + 2^2}} = \frac{4 - 6 + 2}{\sqrt{29}} = \frac{0}{\sqrt{29}} = \boxed{0 \text{ m/s}}$$

11. $\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$, $r = 5$ ft

a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{100\cancel{\pi}}{4\cancel{\pi}\cdot 5^2} = \frac{100}{100} = \boxed{1 \text{ ft/min}}$$

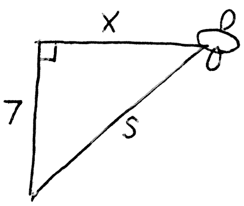
b) $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \cdot 5 \cdot 1 = \boxed{40\pi \text{ ft}^2/\text{min}}$$

12. $\frac{dV}{dt} = k \cdot 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

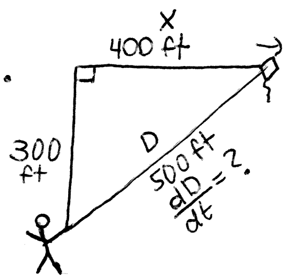
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{k\cdot \cancel{4}\cancel{\pi}\cancel{r^2}}{\cancel{4}\cancel{\pi}\cancel{r^2}} = \boxed{k = \text{constant rate}}$$

13. 

$s = 10$ mi, $\frac{ds}{dt} = 300$ mi/hr $\quad x^2 + 7^2 = s^2 \rightarrow x = \sqrt{s^2 - 49} = \sqrt{10^2 - 49} = \sqrt{51}$

$x^2 + 7^2 = s^2$

$$\cancel{2}x \frac{dx}{dt} = \cancel{2}s \frac{ds}{dt} \rightarrow \frac{dx}{dt} = \frac{s \; ds/dt}{x} = \frac{10 \cdot 300}{\sqrt{51}} = \frac{3000}{\sqrt{51}} \approx \boxed{420.084 \text{ mph}}$$

14. 

$\frac{dx}{dt} = 25$ ft/s

$x^2 + 300^2 = D^2$

$\cancel{2}x \frac{dx}{dt} = \cancel{2}D \frac{dD}{dt}$

$$\frac{dD}{dt} = \frac{x \; dx/dt}{D} = \frac{400(25)}{500} = \frac{100\cancel{00}}{5\cancel{00}} = \boxed{20 \text{ ft/s}}$$

15. $V = \pi r^2 h$, $h = 6$ in, $\frac{dr}{dt} = \frac{0.001 \text{ in}}{3 \text{ min}}$, $\frac{dV}{dt} = ?$ when $r = 1.9$ in

$V = 6\pi r^2$

$$\frac{dV}{dt} = 12\pi r \frac{dr}{dt} = 12 \cdot \pi \cdot 1.9 \cdot \frac{0.001}{3} = \boxed{0.0239 \text{ in}^3/\text{min}}$$

16. $\frac{dV}{dt} = 10 \text{ m}^3/\text{min}$, $V = \frac{1}{3}\pi r^2 h$, $h = \frac{3}{8}d = \frac{3}{8}\cdot 2r = \frac{6}{8}r = \frac{3}{4}r$, $h = 4$ m, $r = 16/3$ m

$\hookrightarrow h = \frac{3}{4}r$ or $r = \frac{4}{3}h$

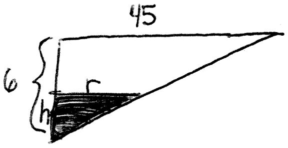
a) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h = \frac{16}{27}\pi h^3$

$$\frac{dV}{dt} = \frac{16}{9}\pi h^2 \frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{dV/dt}{\frac{16}{9}\pi h^2} = \frac{10}{\left(\frac{16}{9}\pi \cdot 4^2\right)} = 0.112 \text{ m/min} = \boxed{11.191 \text{ cm/min}}$$

b) $V = \frac{1}{3}\pi r^2 h = \frac{1}{\cancel{3}}\pi r^2 \cdot \frac{\cancel{3}}{4}r = \frac{\pi}{4}r^3$

$$\frac{dV}{dt} = \frac{3}{4}\pi r^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{dV/dt}{\frac{3}{4}\pi r^2} = \frac{10}{\left(\frac{3}{4}\pi\left(\frac{16}{3}\right)^2\right)} = 0.149 \text{ m/min} = \boxed{14.921 \text{ cm/min}}$$

17. $V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = -50\ m^3/min$, $r = 45\ m$, $h = 6\ m$



$$\frac{45}{6} = \frac{r}{h} \rightarrow 45h = 6r \rightarrow 15h = 2r \rightarrow r = \frac{15}{2}h \text{ or } h = \frac{2}{15}r$$

a) $\frac{dh}{dt} = ?$ when $h = 5\ m$

$$V = \frac{1}{3}\pi\left(\frac{15}{2}h\right)^2 h = \frac{225}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{225}{4}\pi h^2 \frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{dV/dt}{\frac{225}{4}\pi \cdot h^2} = \frac{-50}{\left(\frac{225}{4}\pi \cdot 5^2\right)} = -0.0113\ m/min = \boxed{-1.132\ cm/min}$$

The water level is falling at a rate of 1.132 cm/min.

b) $V = \frac{1}{3}\pi r^2\left(\frac{2}{15}r\right) = \frac{2}{45}\pi r^3$, $r = \frac{15}{2}h = \frac{15}{2} \cdot 5 = \frac{75}{2}\ m$

$$\frac{dV}{dt} = \frac{2}{15}\pi r^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{dV/dt}{\frac{2}{15}\pi r^2} = \frac{-50}{\left(\frac{2}{15}\pi \cdot \left(\frac{75}{2}\right)^2\right)} = -0.0849\ m/min = \boxed{-8.488\ cm/min}$$

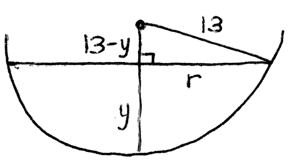
18. $R = 13\ m$, $\frac{dV}{dt} = -6\ m^3/min$

$$V = \frac{\pi}{3}y^2(3R - y) = \frac{\pi}{3}y^2(39 - y) = 13\pi y^2 - \frac{\pi}{3}y^3$$

a) $\frac{dy}{dt} = ?$ when $y = 8m$

$$\frac{dV}{dt} = 26\pi y \frac{dy}{dt} - \pi y^2 \frac{dy}{dt} \rightarrow \frac{dy}{dt} = \frac{dV/dt}{26\pi y - \pi y^2} = \frac{-6}{26\pi(8) - \pi \cdot 8^2} = -0.0133\ m/min = \boxed{-1.326\ cm/min}$$

b)



$$r^2 + (13 - y)^2 = 13^2$$

$$r^2 + \cancel{169} - 26y + y^2 = \cancel{169}$$

$$r^2 = 26y - y^2 \rightarrow \boxed{r = \sqrt{26y - y^2}}$$

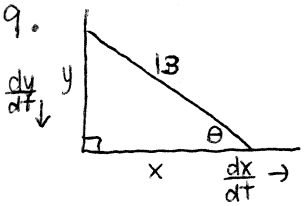
c) $\frac{dr}{dt} = ?$ when $y = 8\ m$

$$r = (26y - y^2)^{1/2}$$

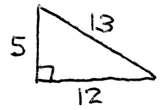
$$\frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-1/2}(26 - 2y)\frac{dy}{dt} = \frac{(13 - y)\ dy/dt}{\sqrt{26y - y^2}} = \frac{(13 - 8)(-0.0133)}{\sqrt{26 \cdot 8 - 8^2}}$$

$$\frac{dr}{dt} = -0.00553\ m/min = \boxed{-0.553\ cm/min}$$

19.



When $x = 12$ ft, $\frac{dx}{dt} = 5$ ft/s



a) $x^2 + y^2 = 13^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \rightarrow \frac{dy}{dt} = \frac{-\cancel{2}x\, dx/dt}{\cancel{2}y} = \frac{-x\, dx/dt}{y} = \frac{-12(\cancel{5})}{\cancel{5}} = \boxed{-12 \text{ ft/s}}$

(down at 12 ft/s)

b) $A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + y\cdot\frac{1}{2}\frac{dx}{dt} = \frac{1}{2}\cdot 12(-12) + 5\cdot\frac{1}{2}\cdot 5 = \frac{-119}{2} = \boxed{-59.5 \text{ ft}^2/\text{s}}$

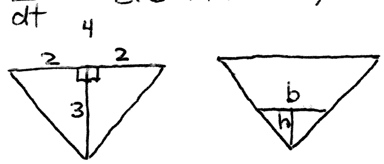
c) $\tan\theta = \frac{y}{x}$

$\sec^2\theta\,\frac{d\theta}{dt} = \frac{x\, dy/dt - y\, dx/dt}{x^2} \rightarrow \frac{d\theta}{dt} = \frac{x\, dy/dt - y\, dx/dt}{x^2\cdot\sec^2\theta}$

$\tan\theta = \frac{5}{12} \rightarrow \theta = \tan^{-1}(5/12)$

$\frac{d\theta}{dt} = \frac{12(-12) - 5(5)}{12^2\cdot\sec^2(\tan^{-1}(5/12))} = \boxed{-1 \text{ radian/s}}$

20. $\frac{dV}{dt} = 2.5$ ft³/min, $l = 15$ ft, $h = 2$ ft

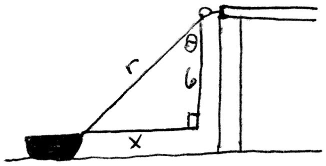


$\frac{4}{3} = \frac{b}{h} \rightarrow 4h = 3b \rightarrow b = \frac{4}{3}h$

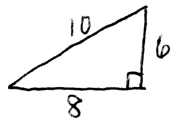
$V = \frac{1}{2}bhl = \frac{1}{2}bh\cdot 15 = \frac{15}{2}\underbrace{b}_{\frac{4}{3}h}h = \frac{15}{2}\cdot\frac{4}{3}h\cdot h = 10h^2$

$V = 10h^2$

$\frac{dV}{dt} = 20h\frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{dV/dt}{20h} = \frac{2.5}{20(2)} = \frac{2.5}{40} = \frac{5}{80} = \boxed{\frac{1}{16} \text{ ft/min}}$

21. 

$\frac{dr}{dt} = -2$ ft/s (distance is becoming shorter)



a) $\frac{dx}{dt} = ?$ when $r = 10$ ft, so $x = 8$ ft

$x^2 + 6^2 = r^2$

$\cancel{2}x \frac{dx}{dt} = \cancel{2}r \frac{dr}{dt}$

$\frac{dx}{dt} = \frac{r\, dr/dt}{x} = \frac{10(-2)}{8} = \frac{-20}{8} = \boxed{-2.5 \text{ ft/s}}$ (approaching the dock)

b) $\tan\theta = \frac{x}{6} \rightarrow \theta = \tan^{-1}\left(\frac{x}{6}\right) = \tan^{-1}\left(\frac{8}{6}\right) = \tan^{-1}\left(\frac{4}{3}\right)$

$\tan\theta = \frac{x}{6}$

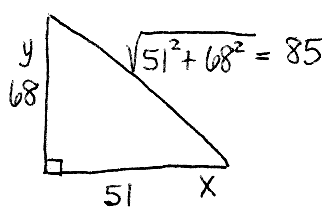
$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{6}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{1/6\, dx/dt}{\sec^2\theta} = \frac{1/6\,(-2.5)}{\sec^2(\tan^{-1}(4/3))} = \boxed{-0.150 \text{ radian/s}}$

22. $\frac{dy}{dt} = 1$ ft/s, $y = 65$ ft, $\frac{dx}{dt} = 17$ ft/s, $\frac{ds}{dt} = ?$ 3 seconds later

$y = 65 + 3(1 \text{ ft/s}) = 68$ ft

$x = 0 + 3(17 \text{ ft/s}) = 51$ ft



$x^2 + y^2 = s^2$

$\cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt} = \cancel{2}s \frac{ds}{dt}$

$\frac{ds}{dt} = \frac{x\, dx/dt + y\, dy/dt}{s} = \frac{51(17) + 68(1)}{85} = \boxed{11 \text{ ft/s}}$