Exponential Growth & Decay (Section 7.4)

* Seperable Differential Equations: $\frac{dy}{dx} = g(x)\cdot h(y)$

STEPS: ① Seperate the x's & the y's. (Put y's on the left & x's on the right)

② Integrate BOTH sides.

③ Solve for y.

④ If given an initial condition, use it to solve for the constant.

ex: $\frac{dy}{dx} = \frac{2y}{x}$

* Only take "y's" to the other side!

$\int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$

$\ln|y| = 2\ln|x| + c$

$e^{\ln|y|} = e^{\ln x^2 + c}$

$y = \cancel{e}^{\cancel{\ln}x^2}\cdot \underbrace{e^c}_{=A}$

$\boxed{y = Ax^2}$

ex: $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$; (1,4)

$\int y^{-1/2}\,dy = \int \boxed{\frac{4\ln x}{x}\,dx}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$2y^{1/2} = \int 4u\,du$

$2y^{1/2} = \frac{4u^2}{2} + c$

$\cancel{2}y^{1/2} = \cancel{2}u^2 + c \longrightarrow y^{1/2} = (\ln x)^2 \pm 2$

@ (1,4) $4^{1/2} = [\ln(1)]^2 + c$

$\pm 2 = c$

$\boxed{y = [(\ln x)^2 \pm 2]^2}$

* Differential Equation: $\frac{dy}{dt} = ky$ w/ an initial condition of $y = y_0$ when $t = 0$.

$\int \frac{1}{y}\,dy = \int k\,dt$

$e^{\ln|y|} = e^{kt + c}$

$y = e^{kt}\cdot \underbrace{e^c}_{=A}$

$y = Ae^{kt} \longrightarrow y = y_0 e^{kt}$

initial amount (y_0); time (t); rate: $k>0 \rightarrow$ growth, $k<0 \rightarrow$ decay

ex: $y(0) = 200$, $y(5) = 16.4$. Write a differential equation that satisfies these conditions.

$y = 200e^{kt}$

$16.4 = 200e^{k(5)}$

$\frac{\ln\left(\frac{16.4}{200}\right)}{5} = \frac{5k}{5}$

$k = -.500$

$\boxed{\therefore\ y = 200e^{-.500t}}$

* Compound Interest: $\underbrace{A(t) = A_0\left(1+\frac{r}{k}\right)^{kt}}_{\text{compounded interest "k" \# of times per year}}$ OR $\underbrace{A(t) = A_0 e^{rt}}_{\text{compounded interest continuously}}$

ex: $A_0 = 5{,}000$
$r = 5\%$
$t = 10$ yrs

compounded monthly -vs- continuously

$A = 5000\left(1+\frac{.05}{12}\right)^{12(10)}$

$\boxed{A = \$8235.05}$

$A = 5000e^{.05(10)}$

$\boxed{A = \$8243.61}$

* Radioactivity: $y = y_0 e^{-kt}$

"k" is always (-) because this is decay.

ex: Polonium-210 decays in t days using:

$y = y_0 e^{-.005t}$

5% left

A sample is not useful after 95% has decayed. How long is a sample good for?

$.05 = e^{-.005t}$

$\frac{\ln(.05)}{-.005} = \frac{-.005t}{-.005}$

$t = 599.146$ $\boxed{\therefore \text{ after 599 days}}$

* Half-Life: $y = y_0 e^{-kt} \longrightarrow \frac{1}{2} = e^{-kt}$ much more useful!

also decay

$\frac{\ln(\frac{1}{2})}{-k} = \frac{-kt}{-k}$

$\frac{-\ln(\frac{1}{2})}{k} = t$

$\boxed{\frac{\ln(2)}{k} = t}$

also useful!

ex: Find the half-life of Polonium-210 in the example above.

$\frac{1}{2} = e^{-.005t}$

$\frac{\ln(\frac{1}{2})}{-.005} = \frac{-.005t}{-.005}$

$\boxed{t = 138.629 \text{ days}}$