Partial Fractions & Logistic Functions (Section 7.5)

* Any rational function can be written as a sum of basic fractions called "Partial Fractions."

** The degree of the numerator must be less than the denominator. Otherwise you MUST do LONG ÷!

ex: $\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)}$

$$\cancel{\frac{2}{(x+1)(x-1)}} = \frac{A(x-1)}{\cancel{(x+1)(x-1)}} + \frac{B(x+1)}{\cancel{(x-1)(x+1)}}$$

$$2 = A(x-1) + B(x+1)$$

let $x=1$: $2 = \cancel{A(1-1)}^{0} + B(1+1)$

$2 = 2B \rightarrow \boxed{B=1}$

let $x=-1$: $2 = A(-1-1) + \cancel{B(-1+1)}^{0}$

$2 = -2A \rightarrow \boxed{A=-1}$

$$\boxed{\frac{2}{x^2-1} = \frac{-1}{x+1} + \frac{1}{x-1}}$$

STEPS

① Factor the denominator

② Set the equation equal to:

$$\frac{A}{\text{factor 1}} + \frac{B}{\text{factor 2}}$$

③ Get a common denominator

④ Let x= the zero of each factor and solve for A & B.

⑤ Write the new answer in Partial fraction form

ex: $\int \left(\frac{8x+1}{x^2-x-6}\right) dx \longrightarrow \int \left(\frac{5}{x-3} + \frac{3}{x+2}\right) dx$

$$\frac{8x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$8x+1 = A(x+2) + B(x-3)$$

let $x=-2$: $8(-2)+1 = A(0) + B(-2-3)$

$-15 = -5B \rightarrow \boxed{B=3}$

let $x=3$: $8(3)+1 = A(3+2) + B(0)$

$25 = 5A \rightarrow \boxed{A=5}$

$= 5\ln|x-3| + 3\ln|x+2| + C$

$$\boxed{= \ln\left|(x-3)^5(x+2)^3\right| + C}$$

#11 $\int \frac{7}{2x^2-5x-3}\,dx \longrightarrow \int\left(\frac{-2}{2x+1}+\frac{1}{x-3}\right)dx$

$$\frac{7}{(2x+1)(x-3)} = \frac{A}{2x+1} + \frac{B}{x-3}$$

$$7 = A(x-3) + B(2x+1)$$

let $x=3$ $\quad 7 = A(0) + B(2\cdot 3+1)$

$$7 = 7B \longrightarrow \boxed{B=1}$$

let $x=-\frac{1}{2}$ $\quad 7 = A(-\frac{1}{2}-3) + B(0)$

$$7 = -\frac{7}{2}A \longrightarrow \boxed{A=-2}$$

$$= \frac{-\cancel{2}\ln|2x+1|}{\cancel{2}} + \ln|x-3| + C$$

$$= -\ln|2x+1| + \ln|x-3| + C$$

$$\boxed{= \ln\left|\frac{x-3}{2x+1}\right| + C}$$

$$\int \frac{-2}{2x+1}\,dx$$

$u = 2x+1$

$du = 2dx$

$$= \int \frac{-1}{u}\,du$$

$$= -\ln|u| + c$$

$$= -\ln|2x+1| + c$$

* Logistic Differential Equations:

Memorize This! → $$\boxed{\frac{dP}{dt} = kP(M-P)}$$

growth constant $\therefore k > 0$

Max Population or "Carrying Capacity"

ex: Example #4 (pg 369)

$\frac{dP}{dt} = .008\,P(100-P)$ *Bears in wildlife preserve per year.

a) What's the carrying capacity? 100 bears

b) When is the pop growing the fastest?

↳ Half the CC or $\frac{M}{2}$ → 50.

c) Rate of change at that point? $\frac{dP}{dt} = .008(50)(100-50)$

$\boxed{= 20 \text{ Bear/year}}$

* General Logistic Formulas:

$\frac{dP}{dt} = kP(M-P)$ ⟶ $$\boxed{P = \frac{M}{1+Ae^{-Mkt}}}$$

Carrying Capacity

Memorize!

growth constant

Rabbits

#31 $P = \frac{1000}{1+e^{4.8-.7t}}$

$e^{4.8} \cdot e^{-.7t}$

$e^{4.8} \downarrow A$ $\quad e^{-.7t} \downarrow e^{-Mkt}$

$-.7t = -Mkt$

$.7 = M \cdot k$

$.7 = 1000\,k$

$\boxed{k = .0007}$

a) $k = .0007$

$M = 1{,}000$

b) $P(0) \approx$ $\boxed{8 \text{ rabbits}}$

$t=0$

$P = \frac{1000}{1+e^{4.8-0}}$

$= \frac{1000}{(1+e^{4.8})}$

$= 8.163$

#30 $\frac{ds}{dt} = \frac{2s+2}{t^2+2t}$; $s(1)=1$

$$\frac{ds}{dt} = \frac{2(s+1)}{t(t+2)}$$

$$\int \frac{1}{s+1}\,ds = \int \frac{2}{t(t+2)}\,dt$$

$$\boxed{\begin{aligned} \frac{2}{t(t+2)} &= \frac{A}{t} + \frac{B}{t+2} \\ 2 &= A(t+2) + B(t) \\ \text{let } t=-2 \quad 2 &= A(0)+B(-2) \rightarrow B=-1 \\ \text{let } t=0 \quad 2 &= A(2)+B(0) \rightarrow A=1 \end{aligned}}$$

$$\int \frac{1}{s+1}\,ds = \int \left(\frac{1}{t} + \frac{-1}{t+2}\right)dt$$

$$\ln|s+1| = \ln|t| - \ln|t+2| + c$$

$$\cancel{e^{\ln}}|s+1| = e^{\ln}\left|\frac{t}{t+2}\right| + c$$

$$s+1 = e^{c}\left(\frac{t}{t+2}\right) = A\left(\frac{t}{t+2}\right) \longrightarrow s+1 = 6\left(\frac{t}{t+2}\right)$$

$$\boxed{s = \frac{6t}{t+2} - 1}$$

$(1, 1)$

s t

$$1+1 = A\left(\frac{1}{3}\right)$$

$$2 = \frac{1}{3}A$$

$$A = 6$$