DATE____________________

8.1–8.5 Concepts Worksheet

NAME____________________

Integration of Rates

One of the basic concepts of integral calculus is that the integral of the rate of change of a quantity can be used to find the net change in that quantity over a period of time. For example, if $f(t)$ is the production rate of an item (in units per month, where t is in months), then $\int_4^{12} f(t)\,dt$ is equal to the number of units produced between $t = 4$ months and $t = 12$ months.

1. Let $R(t)$ represent the rate at which a tapestry is emerging from a weaver's loom. The values of $R(t)$ over a one-hour period are given below.

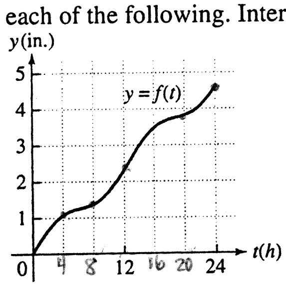
t(min)	0	5	10	15	20	25	30	35	40	45	50	55	60
$R(t)$(cm/min)	10	12	17	20	17	16	15	9	8	12	15	16	13

(a) Write an integral in terms of $R(t)$ that gives the length of tapestry created during this hour. $\int_0^{60} R(t)\,dt$

(b) Use a Trapezoidal Rule approximation to estimate the length of tapestry created during this hour.

$\frac{1}{2}\cdot 5\,(10+2(12)+2(17)+\dots+2(16)+13) = 842.5$ cm

2. Let $r(t)$ represent the rate of rainfall, in inches per hour, t hours after midnight, and let $f(t)$ represent the number of inches of rain that fell during the first t hours after midnight. Use the graph of $y = f(t)$ to estimate each of the following. Interpret your results by giving the real-world meaning of each integral.



(a) $\int_0^{24} r(t)\,dt =$ Inches of rain that fell during the whole day, 4.6 in

(b) $\int_4^{12} r(t)\,dt =$ Inches of rain that fell between 4 AM & noon, $2.3-1.1=1.2$ in

(c) $\int_8^{20} r(t)\,dt =$ Inches of rain that fell between 8 AM & 8 PM, $3.8-1.4=2.4$ in

8.1–8.5 Concepts Worksheet *Continued* NAME________________

3. From the beginning of 2000 to the beginning of 2010, a certain company earned revenues at the rate of $r(t) = t^3 - 15t^2 + 70t + 150$ $\boxed{\text{thousand}}$ dollars per year, where t is the number of years after the beginning of 2000. Find the total revenues for this ten-year period.

$$\int_0^{10} (t^3 - 15t^2 + 70t + 150)\,dt = \left(\tfrac{1}{4}t^4 - 5t^3 + 35t^2 + 150t\right)\Big|_0^{10} = \$2{,}500 \text{ (thousands)}$$

$\$2{,}500{,}000$

Concept Connector

Many rates involve quantities other than time. For example, one might regard force as a rate of work done per unit of distance. In this context, the work formula $W = \int_a^b F(x)\,dx$ can be regarded as another example of a rate being integrated to find the net change of a quantity.

4. A tub weighing 20 pounds is lifted from the ground to a height of 40 feet. As the tub is being lifted, it is also being filled with pebbles so that the weight, in pounds, of the bucket when it is at height x feet is given by $20 + 3x$. Find the amount of work done in lifting the bucket and the pebbles.

$$\int_0^{40} (20 + 3x)\,dx = \left(20x + \tfrac{3}{2}x^2\right)\Big|_0^{40} = 3{,}200$$

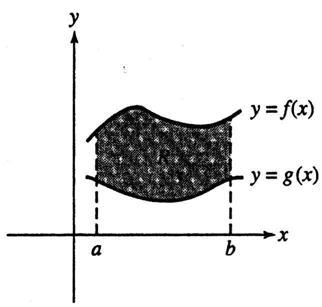
DATE________________________

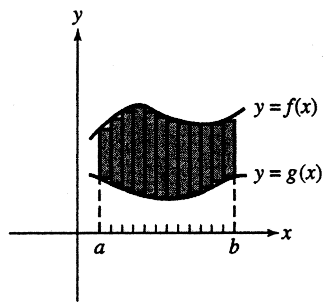
8.2 Concepts Worksheet

NAME________________________

Area as a Definite Integral

To find the area of a region R between curves, one can compute the sum of nonoverlapping rectangular areas that almost fill region R. The integral symbol, $\int$, followed by $f(x)\,dx$, evokes the notion of an infinite sum of rectangular areas having dimensions $f(x)$ and dx. For example, the region R shown below can be approximately filled with rectangles of height $[f(x) - g(x)]$ and width dx, as shown on the right.

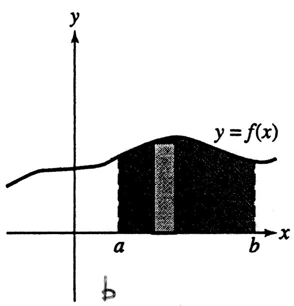




Therefore, the area of region R is: $\int_a^b [f(x) - g(x)]dx$.

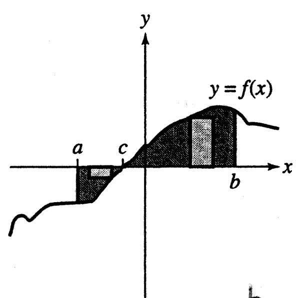
Express the areas of the regions shown using definite integrals without absolute values. Where applicable, use the rectangular section shown as a guide to help you set up the correct integral expression.

1.



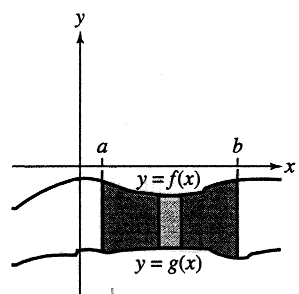
Area: $\int_a^b f(x)dx$

2.



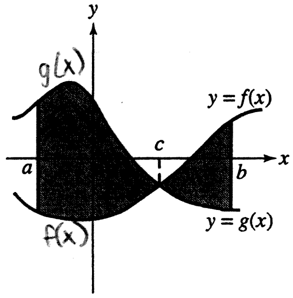
Area: $-\int_a^c f(x)dx + \int_c^b f(x)dx$

3.



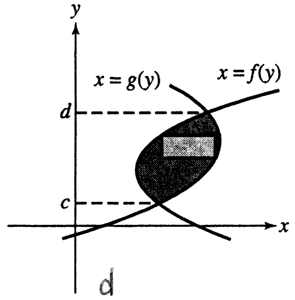
Area: $\int_a^b (f(x) - g(x))dx$

4.



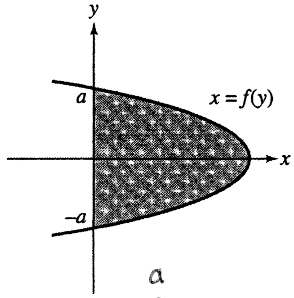
Area: $\int_a^c (g(x) - f(x))dx + \int_c^b (f(x) - g(x))dx$

5.



Area: $\int_c^d (g(y) - f(y))dy$

6.



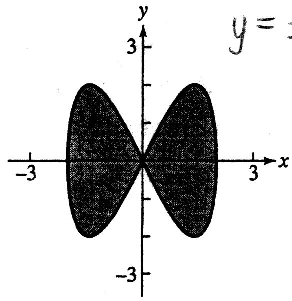
Area: $2\int_0^a f(y)dy$

8.2 Concepts Worksheet Continued NAME________________________

Concept Connectors

7. The graph of $y^2 = 4x^2 - x^4$ is shown below.

$y = \pm\sqrt{4x^2 - x^4}$



$4\int_0^2 \sqrt{4x^2 - x^4}\,dx = 10.667$

Find the area of the shaded region. ________________

8. The national revenue of Urania is modeled by the function $R(t)$ shown below, where t is in years since January 1, 2005, and $R(t)$ is in millions of Uranian Dollars (muds).

$$R(t) = \begin{cases} t & 0 \le t \le 5 \\ 2t - 5 & 5 < t \le 25 \\ t + 20 & 25 < t \end{cases}$$

Urania's domestic expenditures are modeled by the function $E(t)$, also in muds.

$$E(t) = \begin{cases} 3t + 5 & 0 \le t \le 2 \\ 2t + 7 & 2 < t \le 10 \\ \frac{1}{2}t + 22 & 10 < t \end{cases}$$

The annual budget has a surplus if the revenue exceeds expenditures for the year. If expenditures exceed the revenue, the budget runs a deficit. The cumulative surplus/deficit function is the total surplus or deficit that Urania incurs at time t since January 1, 2005.

(a) Describe the cumulative surplus/deficit function $D(t)$ mathematically in terms of R and E.

$D(t) = R(t) - E(t)$, so cumulative $D(t) = \int_0^t (R(x) - E(x))\,dx$

(b) Find the value of the cumulative function for January 1, 2015.

see next page

(c) In what year after 2005 will the value of the cumulative function be 0?

see next page

8b) $\displaystyle\int_0^{10} (R(t) - E(t))\,dt$

$$\int_0^2 (t-3t-5)\,dt = \int_0^2 (-2t-5)\,dt = (-t^2-5t)\Big|_0^2 = -14 \text{ muds 2005 to 2007}$$

$$\int_2^5 (t-2t-7)\,dt = \int_2^5 (-t-7)\,dt = \left(-\tfrac{1}{2}t^2-7t\right)\Big|_2^5 = -31.5 \text{ muds 2007 to 2010}$$

$$\int_5^{10} (\cancel{2t}-5-\cancel{2t}-7)\,dt = \int_5^{10} -12\,dt = -12t\Big|_5^{10} = -60 \text{ muds 2010 to 2015}$$

2005 to 2015 : $-14 - 31.5 - 60 = \boxed{-105.5 \text{ muds}}$

8c) For $t=10$ to $t=25$, $D(t) = 2t-5-\frac{1}{2}t-22 = \frac{3}{2}t-27 = 0$ at $t=18$

Until $t=18$, there is a deficit. After $t=18$, there is a surplus.

$$\int_{10}^{18} \left(2t-5-\tfrac{1}{2}t-22\right)dt = \int_{10}^{18} \left(\tfrac{3}{2}t-27\right)dt = \left(\tfrac{3}{4}t^2-27t\right)\Big|_{10}^{18} = -48 \text{ muds 2015 to 2023}$$

2005 to 2023: $-105.5-48 = -153.5$ muds

Need 153.5 muds surplus to reach a net cumulative value of 0.

$$\int_{18}^{25} \left(2t-5-\tfrac{1}{2}t-22\right)dt = \int_{18}^{25} \left(\tfrac{3}{2}t-27\right)dt = \left(\tfrac{3}{4}t^2-27t\right)\Big|_{18}^{25} = 36.75 \text{ muds 2023 to 2030}$$

2005 to 2030: $-153.5+36.75 = -116.75$ muds (need 116.75 muds surplus)

$$\int_{25}^{x} \left(t+20-\tfrac{1}{2}t-22\right)dt = 116.75 \rightarrow \int_{25}^{x} \left(\tfrac{1}{2}t-2\right)dt = 116.75 \rightarrow \left(\tfrac{1}{4}t^2-2t\right)\Big|_{25}^{x} = 116.75$$

$$\frac{1}{4}x^2 - 2x - 106.25 = 116.75$$

$$\frac{1}{4}x^2 - 2x - 223 = 0$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(\frac{1}{4})(-223)}}{2(1/4)} = \frac{2 \pm \sqrt{227}}{1/2} = 4 \pm 2\sqrt{227}$$

$x = 34.133$ yr ✓

$x = -26.133$ yr X

34.133 yr after 2005 = during $\boxed{2039}$

DATE________________________

8.3 Concepts Worksheet

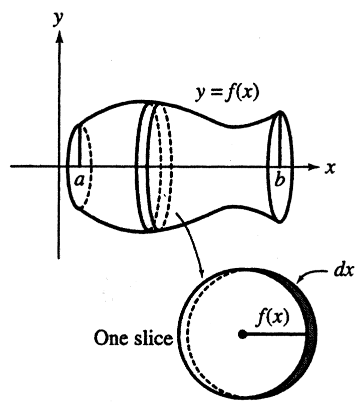
NAME________________________

Volume as a Definite Integral

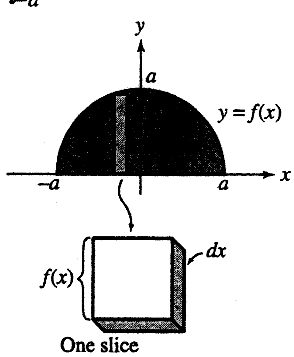
To find the volume of a solid, one can imagine that the solid is made of a stack of thin prisms or cylinders. The volume of each prism or cylinder is equal to the product of the area of a base and the height between the bases. The integral symbol, $\int$, followed by the product of face area and height, where height is an increment dx or dy, represents the volume of the solid.

Note the following examples:

A. Find the volume of the solid generated by revolving the region bounded by the graph of $y = f(x)$, the x-axis, and lines $x = a$ and $x = b$ about the x-axis. Each slice has the shape of a circle whose radius is $f(x)$. The area of this circle is $\pi[f(x)]^2$, so the volume of each cylinder is $\pi[f(x)]^2\,dx$. The volume of the solid is $\pi\int_a^b [f(x)]^2\,dx$.

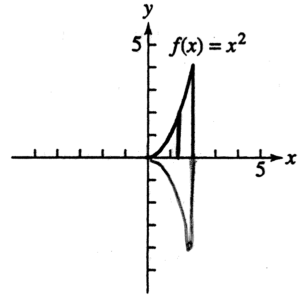


B. Suppose that the base of a solid is the shaded region shown below and that all cross-sections perpendicular to the x-axis are squares. Find the volume of the solid. Each slice has the shape of a square with side length $f(x)$, so the cross sectional area is $[f(x)]^2$ and the volume of each slice is $[f(x)]^2\,dx$. The volume of the solid is $\int_{-a}^{a} [f(x)]^2\,dx$ or, by symmetry, $2\int_0^a [f(x)]^2\,dx$.



8.3 Concepts Worksheet Continued NAME______________________

1. The region bounded by $f(x) = x^2$, the x-axis, and $x = 2$ is rotated about the x-axis.



(a) Draw the solid of revolution on the graph to the right.

(b) Draw and label a slice made perpendicular to the x-axis.

(c) Find the area of a cross section of the solid.

$\pi r^2 = \pi (x^2)^2 = \pi x^4$

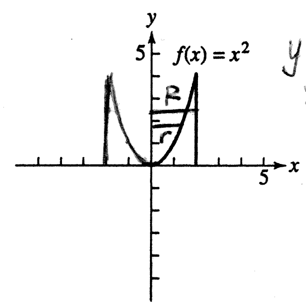
(d) Find the volume of one slice.

$\pi x^4 dx$

(e) Give the volume of the solid as a definite integral. $\int_0^2 \pi x^4 dx$

(f) Evaluate the integral to find the volume of the solid. $\pi \cdot \frac{1}{5} x^5 \Big|_0^2 = \frac{32}{5}\pi$

2. The region bounded by $f(x) = x^2$, the x-axis, and $x = 2$ is rotated about the y-axis.



$y = x^2$
$x = \sqrt{y}$
$R = 2$
$r = \sqrt{y}$

(a) Draw the solid of revolution on the graph to the right.

(b) Draw and label a slice made perpendicular to the y-axis.

(c) Find the area of a cross section of the solid.

$\pi (R^2 - r^2) = \pi (2^2 - \sqrt{y}^2) = \pi (4 - y)$

(d) Find the volume of one slice.

$\pi (4-y)\, dy$

(e) Give the volume of the solid as a definite integral. $\int_0^4 \pi (4-y) dy$

(f) Evaluate the integral to find the volume of the solid.

$\pi (4y - \frac{1}{2} y^2) \Big|_0^4 = \pi (16 - 8) = 8\pi$

8.3 Concepts Worksheet Continued NAME________________

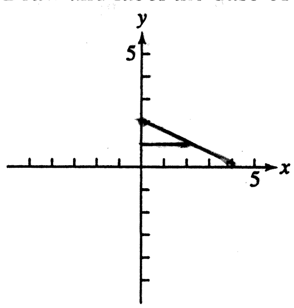
3. The base of a solid is bounded by $x + 2y = 4$ and the coordinate axes. Every cross sectional slice made perpendicular to the y-axis is an isosceles right triangle with the hypotenuse on the base.

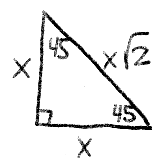
(a) Draw and label the base of the solid.

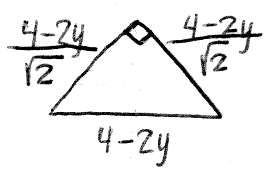
$2y = -x+4$

$y = -\frac{1}{2}x + 2$

$x = 4-2y$







(b) Draw and label a slice made perpendicular to the y-axis.

(c) Find the area of a cross section of the solid. $\frac{1}{2}\left(\frac{4-2y}{\sqrt{2}}\right)\left(\frac{4-2y}{\sqrt{2}}\right) = \frac{(4-2y)^2}{2\cdot 2} = \frac{1}{4}(4-2y)^2$

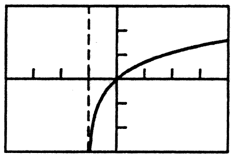
(d) Find the volume of one slice. $\frac{1}{4}(4-2y)^2\,dy$

(e) Give the volume of the solid as a definite integral. $\int_0^2 \frac{1}{4}(4-2y)^2\,dy$

(f) Find the volume of the solid.________________

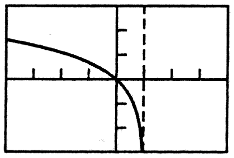
$$\int_0^2 \frac{1}{4}(16-16y+4y^2)\,dy = \int_0^2 (4-4y+y^2)\,dy = \left(4y-2y^2+\frac{1}{3}y^3\right)\Big|_0^2 = \cancel{8}-\cancel{8}+\frac{8}{3} = \frac{8}{3}$$

(b)



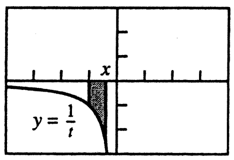
[–4, 4] by [–3, 3]

(c)



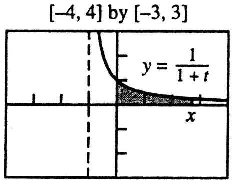
[–4, 4] by [–3, 3]

5. (a)



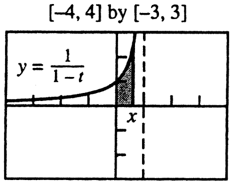
[–4, 4] by [–3, 3]

(b)



[–4, 4] by [–3, 3]

(c)



[–4, 4] by [–3, 3]

6. (a) $\ln |x|$ **(b)** $\ln(1 + x)$

(c) $-\ln(1 - x)$

7. If $t = au$, then $dt = adu$. Therefore, when $t = a$, $u = 1$ and when $t = ax$, $u = x$.

$$\int_a^{ax} \frac{1}{t}\,dt = \int_1^x \frac{1}{au} \cdot adu = \int_1^x \frac{du}{u}$$

$$\int_a^{ax} \frac{1}{t}\,dt = \ln t\Big|_a^{ax} = \ln(ax) - \ln(a)$$

$$\int_1^x \frac{du}{u} = \ln u\Big|_1^x = \ln x - \ln 1 = \ln x$$

Therefore, $\ln(ax) - \ln(a) = \ln x$, so $\ln(ax) = \ln x + \ln(a)$.

8. (a) $\frac{dy}{dx} = \frac{1}{x}$ **(b)** $y = \ln x + C$

9. (a) $\frac{dy}{dx} = \frac{1}{x}$

(b) The equations are the same. Since $\ln 2x = \ln 2 + \ln x$, the general solution $y = \ln x + C$ of the equation $\frac{dy}{dx} = \frac{1}{x}$ includes the particular solution $y = \ln 2x$.

Sections 8.1–8.5

1. (a) $\int_0^{60} R(t)\,dt$ **(b)** 842.5 cm

2. (a) ≈4.6; 4.6 inches of rain fell during the 24 hours beginning at midnight.

(b) ≈1.2; 1.2 inches of rain fell between 4 A.M. and noon.

(c) ≈2.5; 2.5 inches of rain fell between 8 A.M. and 8 P.M.

3. \$2,500,000 **4.** 3200 ft-lb

Section 8.2

1. $\int_a^b f(x)\,dx$

2. $-\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. $\int_a^b [f(x) - g(x)]\,dx$

4. $\int_a^c [g(x) - f(x)]\,dx + \int_c^b [f(x) - g(x)]\,dx$

5. $\int_c^d [g(y) - f(y)]\,dy$

6. $2\int_0^a f(y)\,dy$ or $\int_{-a}^a f(y)\,dy$

7. $\frac{32}{3}$

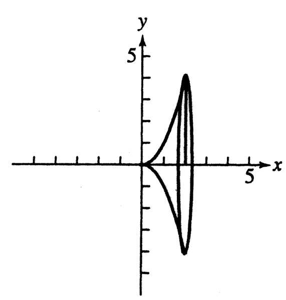
8. (a) $D(t) = \int_0^t (R(x) - E(x))\,dx$

(b) −105.5 or a deficit of 105.5 muds

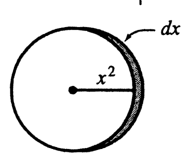
(c) The deficit disappears in 2039.

Section 8.3

1. (a)



(b)



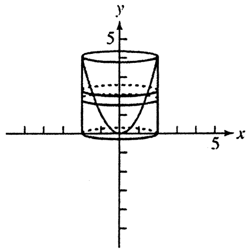
(c) πx^4

(d) $\pi x^4 dx$

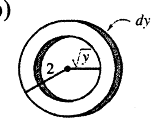
(e) $\pi \int_0^2 x^4 dx$

(f) $\dfrac{32\pi}{5}$

2. (a)



(b)



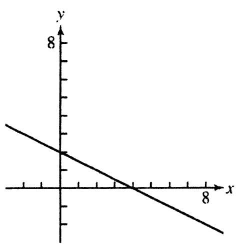
(c) $4\pi - \pi y$

(d) $\pi(4 - y)dy$

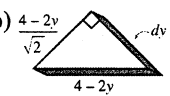
(e) $\pi \int_0^4 (4 - y)\, dy$

(f) 8π

3. (a)



(b)



(c) $\dfrac{1}{2}\left(\dfrac{4-2y}{\sqrt{2}}\right)^2 = (2-y)^2$

(d) $(2-y)^2 dy$

(e) $\int_0^2 (2-y)^2 dy$

(f) $\dfrac{8}{3}$

4. (a) $f(x)$

(b) $2\pi x$

(c) $2\pi x f(x)$

(d) $2\pi x f(x)\, dx$

(e) $2\pi \int_a^b x f(x)\, dx$

Sections 10.1–10.5

1. (a) $-\dfrac{1}{2} < x \le \dfrac{1}{2}$

(b) $f\left(\dfrac{1}{4}\right) = \dfrac{1}{2} - \dfrac{1}{8} + \dfrac{1}{24} - \cdots + \dfrac{(-1)^{n+1}}{n}\left(\dfrac{1}{2}\right)^n + \cdots$

(c) $\ln\left(\dfrac{3}{2}\right)$

(d) $f'(x) = 2 - 4x + 8x^2 - \ldots + (-1)^{n-1}(2)(2x)^{n-1} + \cdots;$

$f'\left(\dfrac{1}{4}\right) = \dfrac{4}{3}$

2. (a) $7 - 3(x-3) + 6(x-3)^2$

(b) 8.44; 0.162

(c) $-2 + 7(x-3) - \dfrac{3}{2}(x-3)^2 + 2(x-3)^3$

(d) $7x^2 - 3x^3 + 6x^4$

3. (a) $f'(x) = 5 - 3x - \dfrac{5}{2!}x^2 + \dfrac{3}{3!}x^3 + \dfrac{5}{4!}x^4 - \dfrac{3}{5!}x^5 - \dfrac{5}{6!}x^6 + \cdots + (-1)^n\left[\dfrac{3}{(2n-1)!}x^{2n-1} + \dfrac{5}{(2n)!}x^{2n}\right] + \cdots$

$f''(x) = -3 - 5x + \dfrac{3}{2!}x^2 + \dfrac{5}{3!}x^3 - \dfrac{3}{4!}x^4 - \dfrac{5}{5!}x^5 + \cdots + (-1)^n\left[\dfrac{3}{(2n-2)!}x^{2n-2} + \dfrac{5}{(2n-1)!}x^{2n-1}\right] + \cdots$

Note that, if n is replaced by $n + 2$ in the general term for $f''(x)$ we obtain the opposite of the general term for $f(x)$. This means that $f''(x) = -f(x)$, so $y = f(x)$ solves $y'' + y = 0$.

(b) All real numbers

(c) $f(x) = 3 \cos x + 5 \sin x$

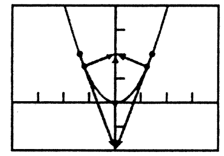
Section 11.2

1. (a) $(-\sqrt{2}, 2), (\sqrt{2}, 2)$

(b) (0, 0)

(c) $(0, 0), \left(\dfrac{\sqrt{6}}{2}, \dfrac{3}{2}\right), \left(-\dfrac{\sqrt{6}}{2}, \dfrac{3}{2}\right)$

(d)



[−4, 4] by [−2, 4]