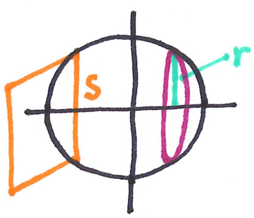
# Volumes of Solids w/ Known Cross Sections (Section 8.3)

* You can use a definite integral of the area of a cross section to find the volume: Volume = Area * width of cross section

$$= \int_a^b \text{Area} \cdot dx \quad \text{or} \quad \int_a^b A(x)\,dx$$

#1 $x^2+y^2=1$




a) cross sections are circles $\rightarrow A=\pi r^2 \rightarrow \pi\left(\sqrt{1-x^2}\right)^2$

$r=y \quad y=\sqrt{1-x^2}$

$\boxed{A=\pi(1-x^2)}$

b) cross sections are squares w/ bases on x-y plane $\rightarrow A=s^2$

$s=\text{chord}=2y$

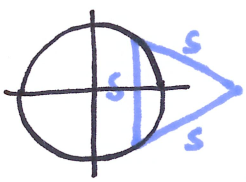
$s=2\sqrt{1-x^2}$

$A=\left(2\sqrt{1-x^2}\right)^2$

$\boxed{A=4(1-x^2)}$

Hard to draw

*c) cross sections are squares w/ diagonals $=\sqrt{2}s$




$\sqrt{2}s=\text{chord}=2y$

$s=\frac{2\sqrt{1-x^2}}{\sqrt{2}}$

$\hookrightarrow A=s^2$

$A=\left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2=\frac{4(1-x^2)}{2}=\boxed{2(1-x^2)}$

d) cross sections are equilateral triangles $\rightarrow A=\frac{\sqrt{3}}{4}\cdot s^2$

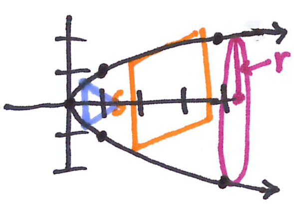
$s=\text{chord}=2y$

$s=2\sqrt{1-x^2}$

$A=\frac{\sqrt{3}}{4}\left(2\sqrt{1-x^2}\right)^2$

$A=\frac{\sqrt{3}}{4}\left[4(1-x^2)\right]=\boxed{\sqrt{3}(1-x^2)}$

#2 $y=\pm\sqrt{x}$




a) circles: $A=\pi r^2=\pi(\sqrt{x})^2=\boxed{\pi x}$

$r=y=\sqrt{x}$

b) squares: $A=s^2=(2\sqrt{x})^2=\boxed{4x}$

$s=2y=2\sqrt{x}$

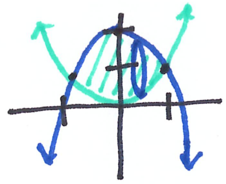
c) squares w/diagonals: $A=s^2=\left(\frac{2\sqrt{x}}{\sqrt{2}}\right)^2=\frac{4x}{2}=\boxed{2x}$

$s=\frac{2y}{\sqrt{2}}=\frac{2\sqrt{x}}{\sqrt{2}}$

d) Equilateral Triangles: $A=\frac{\sqrt{3}}{4}s^2=\frac{\sqrt{3}}{4}(2\sqrt{x})^2=\frac{\sqrt{3}}{4}(4x)=\boxed{\sqrt{3}x}$

$s=2y=2\sqrt{x}$

#4 $y=x^2$; $y=2-x^2$; circles $\rightarrow A=\pi r^2$



$d=$ upper $-$ lower

$(2-x^2)-(x^2)$

$d=2-2x^2$

$r=\frac{2-2x^2}{2}=1-x^2$

$$V=\int_{-1}^{1}\pi(1-x^2)^2\,dx \quad \underline{\underline{\text{OR}}} \quad 2\int_0^1 \pi(1-x^2)^2\,dx \leftarrow$$ This uses symmetry to make it easier!

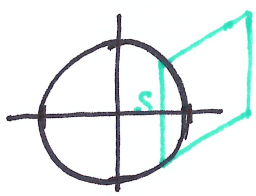
$$=2\pi\int_0^1(1-2x^2+x^4)\,dx$$

$$=2\pi\left(x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5\right)\Big|_0^1$$

$$=2\pi\left(1-\tfrac{2}{3}+\tfrac{1}{5}-0\right)=2\pi\left(\tfrac{8}{15}\right)=\boxed{\frac{16\pi}{15}}$$

#5 $x^2+y^2=1$ ; squares $\rightarrow A=s^2$

$s=2y=2\sqrt{1-x^2}$



$$V=\int_{-1}^{1}\left(2\sqrt{1-x^2}\right)^2dx \quad \underline{\underline{\text{OR}}} \quad 2\int_0^1\left(2\sqrt{1-x^2}\right)^2dx$$

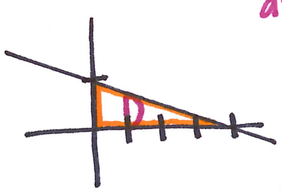
$$=2\int_0^1 4(1-x^2)\,dx$$

$$=8\int_0^1(1-x^2)\,dx$$

$$=8\left(x-\frac{x^3}{3}\right)\Big|_0^1$$

$$=8\left(1-\tfrac{1}{3}-0\right)=8\left(\tfrac{2}{3}\right)=\boxed{\frac{16}{3}}$$

ex: $x+4y=4$; $x=0$; $y=0$; using semi-circles. $\rightarrow A=\frac{\pi}{2}r^2$



$d=y=\frac{4-x}{4}$

$r=\frac{4-x}{4}\cdot\frac{1}{2}=\frac{4-x}{8}$

$$V=\int_0^4\frac{\pi}{2}\left(\frac{4-x}{8}\right)^2dx=\int_0^4\frac{\pi}{128}(4-x)^2dx$$

$$V=\frac{\pi}{128}\int_0^4(16-8x+x^2)\,dx$$

$$=\frac{\pi}{128}\left(16x-4x^2+\frac{x^3}{3}\right)\Big|_0^4$$

$$=\frac{\pi}{128}\left(\cancel{64}-\cancel{64}+\frac{64}{3}-0\right)=\frac{\pi}{\cancel{128}_{2}}\left(\frac{\cancel{64}}{3}\right)=\boxed{\frac{\pi}{6}}$$

ex: same as above except use equil. triangles $\rightarrow A=\frac{s^2\sqrt{3}}{4}$

$s=y=\frac{4-x}{4}$

$$V=\int_0^4\frac{\sqrt{3}\left(\frac{4-x}{4}\right)^2}{4}dx=\int_0^4\frac{\sqrt{3}}{64}(4-x)^2dx$$

$$=\frac{\sqrt{3}}{64}\int_0^4(16-8x+x^2)\,dx$$

$$=\frac{\sqrt{3}}{64}\left(16x-4x^2+\frac{x^3}{3}\right)\Big|_0^4$$

$$=\frac{\sqrt{3}}{64}\left(64-64+\frac{64}{3}-0\right)=\frac{\sqrt{3}}{\cancel{64}}\left(\frac{\cancel{64}}{3}\right)=\boxed{\frac{\sqrt{3}}{3}}$$