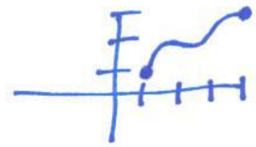
Length of a Curve (Section 8.4)

* To find the length of a straight line, you use the distance formula: $\text{distance} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

* What if the line is not straight?



In Calculus, we add up a bunch of straight lines.

$$d = \sqrt{(\underbrace{x_2-x_1}_{\text{change of } x})^2+(\underbrace{y_2-y_1}_{\text{change of } y})^2} = \sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{(dx)^2+(dy)^2}$$

* Now do some Algebra on that:

$$\sqrt{\left[(dx)^2+(dy)^2\right]\cdot\left(\frac{dx}{dx}\right)^2} = \sqrt{\left[\left(\frac{dx}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\right]\cdot(dx)^2}$$

$$= \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right](dx)^2}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

* Length of a Smoothe Curve:

All derivatives in the interval MUST exist.

*If not, split the integral where the derivative does not exist or solve for the variable.

$$L = \boxed{\int_a^b}\sqrt{1+\left(\overset{y'}{\frac{dy}{dx}}\right)^2}\,dx \quad \text{OR} \quad \boxed{\int_c^d}\sqrt{1+\left(\overset{x'}{\frac{dx}{dy}}\right)^2}\,dy$$

x-values   y-values

ex: $y=\tan x \quad -\pi/3 \le x \le 0$ Find the length of the curve.

$y' = \frac{dy}{dx} = \sec^2 x$

$$L = \int_{-\pi/3}^{0}\sqrt{1+(\sec^2 x)^2}\,dx = \int_{-\pi/3}^{0}\sqrt{1+\sec^4 x}\,dx$$

* Use calculator!

$\boxed{\approx 2.057}$

ex: $x = \frac{1}{4}y^4 + \frac{1}{8}y^{-2}$ from $y=1$ to $y=3$ OR $[1,3]$

$x' = \frac{dx}{dy} = y^3 - \frac{1}{4}y^{-3}$

$$L = \int_1^3 \sqrt{1 + (y^3 - \tfrac{1}{4}y^{-3})^2}\, dy$$

*To do this by hand, you must SIMPLIFY!

$1 + y^6 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16}y^{-6}$

$y^6 + 1/2 + 1/16\, y^{-6}$

$(y^3 + \frac{1}{4}y^{-3})^2$

$$= \int_1^3 \cancel{\sqrt{\cancel{(y^3 + \tfrac{1}{4}y^{-3})^2}}}\, dy$$

$$= \int_1^3 \left(y^3 + \tfrac{1}{4}y^{-3}\right) dy$$

*Now its a simple polynom. integration!

$\boxed{\approx 20.111}$

#19 $L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx$ @ $(1,1)$

a) $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} \rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\int dy = \int \frac{1}{2} x^{-1/2} dx$

$y = \frac{1}{2} x^{1/2} \cdot 2 + c$

* use initial cond.

$1 = (1)^{1/2} + c$

$c = 0 \quad \therefore \boxed{y = x^{1/2}}$

b) 2 curves: when you square root on the first step you get +/−.

#27 $y = x^3 + 5|x|$ $[-2,1]$ *The der

$$y = \begin{cases} x^3 - 5x & -2 \le x \le 0 \\ x^3 + 5x & 0 < x \le 1 \end{cases} \qquad y' = \begin{cases} 3x^2 - 5 & -2 \le x \le 0 \\ 3x^2 + 5 & 0 < x \le 1 \end{cases}$$

$$\therefore L = \underbrace{\int_{-2}^{0} \sqrt{1 + (3x^2 - 5)^2}\, dx}_{7.0432\ldots} + \underbrace{\int_0^1 \sqrt{1 + (3x^2+5)^2}\, dx}_{6.08444\ldots}$$

$7.0432\ldots + 6.08444\ldots$

$\boxed{\approx 13.132}$