Section 9.3: 1-45 e.o.o., 39

1. e^x vs. x^3-3x+1

$$\lim_{x\to\infty}\frac{e^x}{x^3-3x+1}=\frac{\infty}{\infty}\to\lim_{x\to\infty}\frac{e^x}{3x^2-3}=\frac{\infty}{\infty}\to\lim_{x\to\infty}\frac{e^x}{6x}=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{e^x}{6}=\frac{\infty}{6}=\boxed{\infty}$$, so $\boxed{\begin{array}{l} e^x\text{: faster} \\ x^3-3x+1\text{: slower}\end{array}}$

5. $\ln x$ vs. $x-\ln x$

$$\lim_{x\to\infty}\frac{\ln x}{x-\ln x}\to\lim_{x\to\infty}\frac{1/x}{1-1/x}=\frac{1/\infty}{1-1/\infty}=\frac{0}{1-0}=\frac{0}{1}=\boxed{0}$$, so $\boxed{\begin{array}{l}\ln x\text{: slower} \\ x-\ln x\text{: faster}\end{array}}$

9. x^2 vs. x^2+4x

$$\lim_{x\to\infty}\frac{x^2}{x^2+4x}=\frac{\infty}{\infty}\to\lim_{x\to\infty}\frac{2x}{2x+4}=\frac{\infty}{\infty}\to\lim_{x\to\infty}\frac{2}{2}=\boxed{1}$$, so $\boxed{x^2\ \&\ x^2+4\text{: same}}$

13. $\ln x$ vs. $\log\sqrt{x}$

$$\log\sqrt{x}=\log x^{1/2}=\frac{1}{2}\log x=\frac{1}{2}\log_{10}x=\frac{1}{2}\left(\frac{\ln x}{\ln 10}\right)=\frac{1}{2\ln 10}\cdot\ln x$$

$$\lim_{x\to\infty}\frac{\log\sqrt{x}}{\ln x}=\lim_{x\to\infty}\frac{\frac{1}{2\ln 10}\cdot\cancel{\ln x}}{\cancel{\ln x}}=\boxed{\frac{1}{2\ln 10}}$$, so $\boxed{\ln x\ \&\ \log\sqrt{x}\text{: same}}$

17. e^x vs. $x\ln x-x$

$$\lim_{x\to\infty}\frac{x\ln x-x}{e^x}\to\lim_{x\to\infty}\frac{\cancel{x}\cdot\frac{1}{\cancel{x}}+\ln x\cdot 1-1}{e^x}=\lim_{x\to\infty}\frac{\cancel{1}+\ln x-\cancel{1}}{e^x}=\lim_{x\to\infty}\frac{\ln x}{e^x}=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{1/x}{e^x}=\lim_{x\to\infty}\frac{1}{xe^x}=\frac{1}{\infty e^\infty}=\frac{1}{\infty}=\boxed{0}$$, so $\boxed{\begin{array}{l}x\ln x-x\text{: slower} \\ e^x\text{: faster}\end{array}}$

21. x^2 vs. x^3+3

$$\lim_{x\to\infty}\frac{x^2}{x^3+3}=\frac{\infty}{\infty}\to\lim_{x\to\infty}\frac{2x}{3x^2}=\lim_{x\to\infty}\frac{2}{3x}=\frac{2}{3\infty}=\boxed{0}$$, so $\boxed{\begin{array}{l}x^2\text{: slower} \\ x^3+3\text{: faster}\end{array}}$

25. $\ln x$ vs. $\log_2 x^2$

$$\log_2 x^2=2\log_2 x=2\left(\frac{\ln x}{\ln 2}\right)=\frac{2}{\ln 2}\cdot\ln x$$

$$\lim_{x\to\infty}\frac{\log_2 x^2}{\ln x}=\lim_{x\to\infty}\frac{\frac{2}{\ln 2}\cdot\cancel{\ln x}}{\cancel{\ln x}}=\boxed{\frac{2}{\ln 2}}$$, so $\boxed{\ln x\ \&\ \log_2 x^2\text{: same}}$

29. $e^x, x^x, (\ln x)^x, e^{x/2}$

$$\lim_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim_{x\to\infty} e^x - e^{\frac{1}{2}x} = \lim_{x\to\infty} e^{\frac{1}{2}x} = e^{\frac{1}{2}\infty} = \boxed{\infty} \rightarrow$$ e^x: faster, $e^{x/2}$: slower $\quad e^{x/2} < e^x$

$$\lim_{x\to\infty} \frac{(\ln x)^x}{x^x} = \lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^x \rightarrow \lim_{x\to\infty} \left(\frac{1/x}{1}\right)^x = \left(\frac{0}{1}\right)^\infty = 0^\infty = \boxed{0} \rightarrow$$ $(\ln x)^x$: slower, x^x: faster $\quad (\ln x)^x < x^x$

$$\lim_{x\to\infty} \frac{(\ln x)^x}{e^x} = \lim_{x\to\infty} \left(\frac{\ln x}{e}\right)^x = \left(\frac{\infty}{e}\right)^\infty = \infty^\infty = \boxed{\infty} \rightarrow$$ $(\ln x)^x$: faster, e^x: slower $\quad e^x < (\ln x)^x$

$$\boxed{e^{x/2} < e^x < (\ln x)^x, x^x}$$

33. $f_1(x) = 3^x, \; f_2(x) = \sqrt{9^x+2^x}, \; f_3(x) = \sqrt{9^x-4^x}$

$f_1(x)$ & $f_2(x)$:

$$\lim_{x\to\infty} \frac{\sqrt{9^x+2^x}}{3^x} = \lim_{x\to\infty} \frac{\sqrt{9^x+2^x}}{\sqrt{(3^x)^2}} = \lim_{x\to\infty} \sqrt{\frac{9^x+2^x}{(3^2)^x}} = \lim_{x\to\infty} \sqrt{\frac{9^x+2^x}{9^x}}$$

$$\lim_{x\to\infty} \sqrt{\frac{9^x}{9^x} + \frac{2^x}{9^x}} = \lim_{x\to\infty} \sqrt{1+\left(\frac{2}{9}\right)^x} = \sqrt{1+\left(\frac{2}{9}\right)^\infty} = \sqrt{1+0} = \sqrt{1} = \boxed{1}$$, so $\boxed{f_1 \; \& \; f_2 \text{: same}}$

$f_1(x)$ & $f_3(x)$:

$$\lim_{x\to\infty} \frac{\sqrt{9^x-4^x}}{3^x} = \lim_{x\to\infty} \frac{\sqrt{9^x-4^x}}{\sqrt{(3^x)^2}} = \lim_{x\to\infty} \sqrt{\frac{9^x-4^x}{(3^2)^x}} = \lim_{x\to\infty} \sqrt{\frac{9^x-4^x}{9^x}}$$

$$\lim_{x\to\infty} \sqrt{\frac{9^x}{9^x} - \frac{4^x}{9^x}} = \lim_{x\to\infty} \sqrt{1-\left(\frac{4}{9}\right)^x} = \sqrt{1-\left(\frac{4}{9}\right)^\infty} = \sqrt{1-0} = \sqrt{1} = \boxed{1}$$, so $\boxed{f_1 \; \& \; f_3 \text{: same}}$

If f_1 & f_2 grow at the same rate, and f_1 & f_3 grow at the same rate, then f_2 & f_3 must also grow at the same rate.

37. $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ = constant (non-zero), so f & g grow at the $\boxed{\text{same}}$ rate.

39. a) $\lim_{x\to\infty} \frac{e^x}{x^n} = \frac{\infty}{\infty}$, but the numerator will always be e^x, while eventually the denominator will be a constant.

$$\lim_{x\to\infty} \frac{e^x}{\text{constant}} = \frac{\infty}{c} = \boxed{\infty}$$, so $\boxed{e^x\text{: faster } \& \; x^n\text{: slower}}$

39. b) $\lim\limits_{x\to\infty} \frac{a^x}{x^n} = \frac{\infty}{\infty}$, but the numerator will always be a power, while eventually the denominator will be a constant.

$$\lim_{x\to\infty} \frac{a^x}{\text{constant}} = \frac{\infty}{c} = \boxed{\infty}, \text{ so } \boxed{a^x\text{: faster } \& \ x^n\text{: slower}}$$

41. a) $\lim\limits_{x\to\infty} \frac{\ln x}{x^{1/n}} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{1/x}{\frac{1}{n}\cdot x^{\frac{1}{n}-1}} = \lim_{x\to\infty} \frac{nx^{-1}}{x^{\frac{1}{n}-1}} = \lim_{x\to\infty} nx^{-1-(\frac{1}{n}-1)} = \lim_{x\to\infty} nx^{-1/n} = \lim_{x\to\infty} \frac{n}{x^{1/n}}$$

$$\lim_{x\to\infty} \frac{n}{x^{1/n}} = \frac{n}{\infty^{1/n}} = \frac{n}{\infty} = \boxed{0}, \text{ so } \boxed{\ln x\text{: slower } \& \ x^{1/n}\text{: faster}}$$

b) $\lim\limits_{x\to\infty} \frac{\ln x}{x^a} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{1/x}{ax^{a-1}} = \lim_{x\to\infty} \frac{x^{-1}}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{a}\cdot x^{-1-(a-1)} = \lim_{x\to\infty} \frac{1}{a}\cdot x^{-a} = \lim_{x\to\infty} \frac{1}{ax^a}$$

$$\lim_{x\to\infty} \frac{1}{ax^a} = \frac{1}{a\cdot\infty^a} = \frac{1}{\infty} = \boxed{0}, \text{ so } \boxed{\ln x\text{: slower } \& \ x^a\text{: faster}}$$

45. If polynomials $p(x)$ and $q(x)$ grow at the same rate, then $\lim\limits_{x\to\pm\infty} \frac{p(x)}{q(x)}$ is a constant number. If the polynomials grow at the same rate, then they must be the same degree; otherwise $\lim\limits_{x\to\pm\infty} \frac{p(x)}{q(x)}$ would be ∞ or 0. $\lim\limits_{x\to\pm\infty} \frac{p(x)}{q(x)}$ is equivalent to the ratio of the leading coefficients.

For example, $\lim\limits_{x\to\infty} \frac{2x^2+x-3}{5x^2+13x+100} = \lim\limits_{x\to\infty} \frac{2\cancel{x^2}}{5\cancel{x^2}} = \frac{2}{5}$.