Section 9.4: 1–45 e.o.o.

1. $\displaystyle\int_0^\infty \frac{2x}{x^2+1}\,dx = \boxed{\lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}\,dx}$

$\displaystyle\lim_{b\to\infty} \ln(x^2+1)\Big|_0^b = \lim_{b\to\infty}\left(\ln(b^2+1) - \cancelto{0}{\ln(1)}\right) = \lim_{b\to\infty}\ln(b^2+1) = \ln(\infty^2+1) = \boxed{\infty}$

5. $\displaystyle\int_1^\infty x^{-4}\,dx = \lim_{b\to\infty}\int_1^b x^{-4}\,dx = \lim_{b\to\infty} -\frac{1}{3}x^{-3}\Big|_1^b = \lim_{b\to\infty}\frac{-1}{3x^3}\Big|_1^b$

$\displaystyle\lim_{b\to\infty}\left(\frac{-1}{3b^3}+\frac{1}{3}\right) = \frac{-1}{3\infty^3}+\frac{1}{3} = 0+\frac{1}{3} = \boxed{\frac{1}{3}}$

9. $\displaystyle\int_{-\infty}^{-1} x^{-2}\,dx = \lim_{a\to-\infty}\int_a^{-1} x^{-2}\,dx = \lim_{a\to-\infty} -x^{-1}\Big|_a^{-1} = \lim_{a\to-\infty}\frac{-1}{x}\Big|_a^{-1}$

$\displaystyle\lim_{a\to-\infty}\left(\frac{-1}{-1}+\frac{1}{a}\right) = 1+\frac{1}{-\infty} = 1+0 = \boxed{1}$

3. $\displaystyle\int_{-1}^\infty \frac{1}{x^2+5x+6}\,dx \qquad \frac{1}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3}$

$A(x+3)+B(x+2) = 1$

$x=-3: \quad -B = 1 \to B = -1$

$x=-2: \quad A = 1$

$\displaystyle\lim_{b\to\infty}\int_{-1}^b\left(\frac{1}{x+2}-\frac{1}{x+3}\right)dx = \lim_{b\to\infty}\left(\ln|x+2|-\ln|x+3|\right)\Big|_{-1}^b = \lim_{b\to\infty}\ln\left|\frac{x+2}{x+3}\right|\Bigg|_{-1}^b$

$\displaystyle\lim_{b\to\infty}\left(\ln\left|\frac{b+2}{b+3}\right| - \ln\left|\frac{1}{2}\right|\right) = \lim_{b\to\infty}\left(\ln\left|\frac{1+2/b}{1+3/b}\right| - \ln(1/2)\right)$

$\displaystyle\ln\left|\frac{1+\frac{2}{\infty}}{1+\frac{3}{\infty}}\right| - \ln(1/2) = \ln\left|\frac{1+0}{1+0}\right| - \ln\left(\frac{1}{2}\right) = \cancelto{0}{\ln(1)} - \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)^{-1} = \boxed{\ln 2}$

17. $$\int_1^{\infty} xe^{-2x}\,dx = \lim_{b\to\infty}\int_1^{b} xe^{-2x}\,dx$$

Tabular integration:

x		e^{-2x}
x	$+$	e^{-2x}
1	$-$	$-\frac{1}{2}e^{-2x}$
0		$\frac{1}{4}e^{-2x}$

$$\lim_{b\to\infty}\left(\frac{-1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\right)\Big|_1^b = \lim_{b\to\infty}\left[\left(-\frac{1}{2}be^{-2b}-\frac{1}{4}e^{-2b}\right)-\left(\frac{-1}{2}e^{-2}-\frac{1}{4}e^{-2}\right)\right]$$

$$\lim_{b\to\infty}\left[\left(\frac{-b}{2e^{2b}}-\frac{1}{4e^{2b}}\right)-\left(\frac{-1}{2e^2}-\frac{1}{4e^2}\right)\right] = \frac{-1}{4e^{2\infty}}-\frac{1}{4e^{2\infty}}+\frac{1}{2e^2}+\frac{1}{4e^2}$$

$$= 0+0+\frac{1}{2e^2}+\frac{1}{4e^2} = \frac{2}{4e^2}+\frac{1}{4e^2} = \boxed{\frac{3}{4e^2}}$$

21. $$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \lim_{a\to-\infty}\int_a^0 e^x\,dx + \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$\lim_{a\to-\infty} e^x\Big|_a^0 + \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{a\to-\infty}\left(e^0-e^a\right)+\lim_{b\to\infty}\left(\frac{-1}{e^b}+\frac{1}{e^0}\right)$$

$$\lim_{a\to-\infty}\left(1-e^a\right)+\lim_{b\to\infty}\left(\frac{-1}{e^b}+1\right) = 1-e^{-\infty}+\frac{-1}{e^{\infty}}+1$$

$$= 1-\frac{1}{e^{\infty}}-\frac{1}{e^{\infty}}+1 = 1-0-0+1 = \boxed{2}$$

25. a) The integral is improper because there is a vertical asymptote (infinite discontinuity) when $x=1$.

b) $$\int_0^2 \frac{1}{1-x^2}\,dx \qquad \frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} = \frac{A}{1+x}+\frac{B}{1-x}$$

$$A(1-x)+B(1+x) = 1$$

$x=1$: $2B = 1 \rightarrow B = 1/2$

$x=-1$: $2A = 1 \rightarrow A = 1/2$

$$\int_0^2 \frac{1}{1-x^2}\,dx = \int_0^2\left(\frac{1/2}{1+x}+\frac{1/2}{1-x}\right)dx = \frac{1}{2}\int_0^2\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

25. b) (continued)

$$\lim_{b\to 1^-} \frac{1}{2}\int_0^b \left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx + \lim_{a\to 1^+} \frac{1}{2}\int_a^2 \left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$\lim_{b\to 1^-} \frac{1}{2}\left(\ln|1+x|+\ln|1-x|\right)\Big|_0^b + \lim_{a\to 1^+} \frac{1}{2}\left(\ln|1+x|+\ln|1-x|\right)\Big|_a^2$$

$$\lim_{b\to 1^-} \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Bigg|_0^b + \lim_{a\to 1^+} \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Bigg|_a^2$$

$$\lim_{b\to 1^-}\left(\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right| - \overset{0}{\cancel{\frac{1}{2}\ln\left|\frac{1}{1}\right|}}\right) + \lim_{a\to 1^+}\left(\frac{1}{2}\ln\left|\frac{3}{-1}\right| - \frac{1}{2}\ln\left|\frac{1+a}{1-a}\right|\right)$$

$$\lim_{b\to 1^-} \frac{1}{2}\ln\left|\frac{1/b+1}{1/b-1}\right| + \frac{1}{2}\ln 3 - \lim_{a\to 1^+}\frac{1}{2}\ln\left|\frac{1/a+1}{1/a-1}\right|$$

$$\frac{1}{2}\ln\left|\frac{1+1}{1-1}\right| + \frac{1}{2}\ln 3 - \frac{1}{2}\ln\left|\frac{1+1}{1-1}\right| = \frac{1}{2}\ln\left|\frac{2}{0}\right| + \frac{1}{2}\ln 3 - \frac{1}{2}\ln\left|\frac{2}{0}\right|$$

$$\frac{1}{2}\ln\infty + \frac{1}{2}\ln 3 - \frac{1}{2}\ln\infty \to \boxed{\text{Diverges}}$$

29. a) The integral is improper because there is a vertical asymptote (infinite discontinuity) at $x=0$.

b) $$\int_0^1 x\ln x\,dx = \lim_{a\to 0^+}\int_a^1 \underbrace{x}_{} \overbrace{\underbrace{\ln x}_{u}\;dx}^{dv}$$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$uv - \int v\,du = \lim_{a\to 0^+}\left(\frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx\right)\Big|_a^1$$

$$\lim_{a\to 0^+}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_a^1 = \lim_{a\to 0^+}\left[\left(\overset{0}{\cancel{\tfrac{1}{2}\ln 1}} - \frac{1}{4}\right) - \left(\frac{1}{2}a^2\ln a - \frac{1}{4}a^2\right)\right]$$

$$\lim_{a\to 0^+}\left(\frac{-1}{4} - \frac{1}{2}a^2\ln a + \frac{1}{4}a^2\right) = \frac{-1}{4} - 0 + 0 = \boxed{\frac{-1}{4}}$$

33. $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$ $\qquad \underset{+2}{-1} \le \underset{+2}{\cos x} \le \underset{+2}{1}$, so $1 \le 2+\cos x \le 3$

$$\frac{1}{x} \le \frac{2+\cos x}{x} \le \frac{3}{x}$$

$$\int_{\pi}^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_{\pi}^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x| \Big|_{\pi}^{b} = \lim_{b\to\infty} (\ln b - \ln \pi) = \ln\infty - \ln\pi$$
$$= \infty - \ln\pi = \infty$$

$\displaystyle\int_{\pi}^{\infty} \frac{1}{x}\,dx$ diverges. Since $\frac{1}{x} \le \frac{2+\cos x}{x}$, $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$ also $\boxed{\text{diverges}}$.

37. $\displaystyle\int_{0}^{\infty} \frac{1}{(1+s)\sqrt{s}}\,ds$ has a vertical asymptote (infinite discontinuity) at $x=0$.

$$\int_{0}^{\infty} \frac{1}{(1+\sqrt{s}^{2})\sqrt{s}}\,ds = \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{(1+\sqrt{s}^{2})\sqrt{s}}\,ds + \lim_{b\to\infty} \int_{1}^{\infty} \frac{1}{(1+\sqrt{s}^{2})\sqrt{s}}\,ds$$

$$\lim_{a\to 0^+} 2\tan^{-1}\sqrt{s}\Big|_{a}^{1} + \lim_{b\to\infty} 2\tan^{-1}\sqrt{s}\Big|_{1}^{b}$$

$$\lim_{a\to 0^+} (2\tan^{-1}1 - 2\tan^{-1}\sqrt{a}) + \lim_{b\to\infty} (2\tan^{-1}\sqrt{b} - 2\tan^{-1}1)$$

$$\cancel{2\cdot\frac{\pi}{4}} - 2\tan^{-1}0 + 2\tan^{-1}\infty - \cancel{2\cdot\frac{\pi}{4}} = -2(0) + \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) = 0 + \pi = \boxed{\pi}$$

41. $\int_0^2 \frac{1}{1-t}\,dt$ has a vertical asymptote (infinite discontinuity) at $x=1$.

$$\lim_{b\to 1^-}\int_0^b \frac{1}{1-t}\,dt + \lim_{a\to 1^+}\int_a^2 \frac{1}{1-t}\,dt = \lim_{b\to 1^-} -\ln|1-t|\Big|_0^b + \lim_{a\to 1^+} -\ln|1-t|\Big|_a^2$$

$$\lim_{b\to 1^-}\left(-\ln|1-b| + \cancel{\ln 1}\right) + \lim_{a\to 1^+}\left(\cancel{-\ln 1} + \ln|1-a|\right) = -\ln 0 + \ln 0 = -(-\infty) + -\infty$$

$\boxed{\text{Diverges}}$

45. We're not going to do this problem.
If you made it to this part of the assignment,
give yourself a free point!