Improper Integrals (Section 9.4)

* Integrals with infinite limits of integration are called improper integrals.

1) If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3) If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx}$$ (where c is any real number)

* the integral converges only if both integrals have a value; otherwise it diverges.

ex: $$\int_{-\infty}^{\infty} e^x\,dx = \lim_{a\to-\infty}\int_a^0 e^x\,dx + \lim_{b\to\infty}\int_0^b e^x\,dx$$

$$= \lim_{a\to-\infty} e^x\Big|_a^0 + \lim_{b\to\infty} e^x\Big|_0^b$$

$$= \lim_{a\to-\infty}\left(e^0 - e^a\right) + \lim_{b\to\infty}\left(e^b - e^0\right)$$

$$\underbrace{1-0}_{\text{converge}} + \underbrace{\infty - 1}_{\text{diverge}} = \infty \therefore \text{Diverges!}$$

ex: $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b$

$= \lim_{b\to\infty} (\ln b - \ln 1) = \infty \therefore$ Diverges!

ex: $\int_0^{\infty} \frac{2}{x^2+4x+3}\,dx = \lim_{b\to\infty} \int_0^b \underbrace{\frac{2}{x^2+4x+3}}_{\text{Partial Fractions}}\,dx$

$\frac{2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$2 = Ax + 3A + Bx + B$

$0 = A + B$
$2 = 3A + B$

$2 = 3A + \cancel{B}$
$0 = -A - \cancel{B}$
$2 = 2A$
$A = 1$
$B = -1$

$= \lim_{b\to\infty} \int_0^b \left(\frac{1}{x+1} + \frac{-1}{x+3}\right) dx$

$= \lim_{b\to\infty} \left[\ln(x+1)\Big|_0^b - \ln(x+3)\Big|_0^b\right]$

$= \lim_{b\to\infty} \left[\ln(b+1) - \cancel{\ln 1}^{0} - (\ln(b+3) - \ln 3)\right]$

$= \lim_{b\to\infty} \left[\ln\left(\frac{b+1}{b+3}\right) + \ln 3\right]$

$= \lim_{b\to\infty} \left[\ln\left(\frac{1+1/b}{1+3/b}\right) + \ln 3\right] = \ln(1) + \ln 3 = \boxed{\ln 3}$

ex: $\int_1^{\infty} x \ln x\,dx = \lim_{b\to\infty} \int_1^b x \ln x\,dx$

$u = \ln x \qquad dv = x\,dx$
$du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$

$= \lim_{b\to\infty} \left[\frac{x^2}{2} \ln x \Big|_1^b - \int_1^b \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}}\,dx\right]$

$= \lim_{b\to\infty} \left[\frac{x^2}{2} \ln x \Big|_1^b - \frac{x^2}{4}\Big|_1^b\right]$

$= \lim_{b\to\infty} \left[\frac{b^2}{2} \ln b - \frac{1}{2}\cancel{\ln 1}^{0} - \left(\frac{b^2}{4} - \frac{1}{4}\right)\right] = \infty \therefore$ Diverges!

ex: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} + \int_0^{\infty} = \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$

$= \lim_{a\to-\infty} \left(\tan^{-1} x \Big|_a^0\right) + \lim_{b\to\infty} \left(\tan^{-1} x \Big|_0^b\right)$

$= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) + \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0)$

$0 - (-\pi/2) \quad + \quad \pi/2 - 0 \quad \boxed{= \pi}$

* Integrands w/ infinite discontinuites: you have to break up the integral. Both parts must converge for the integral to converge.

ex: $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 + \int_1^3$

$$= \lim_{c\to 1^-} \int_0^c \frac{1}{(x-1)^{2/3}}\,dx + \lim_{c\to 1^+} \int_c^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{c\to 1^-} 3(x-1)^{1/3}\Big|_0^c + \lim_{c\to 1^+} 3(x-1)^{1/3}\Big|_c^3$$

$$= \lim_{c\to 1^-}\left[3(c-1)^{1/3} - 3(0-1)^{1/3}\right] + \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right]$$

$$\underbrace{0 + 3}_{\text{conv.}} \qquad \underbrace{3\sqrt[3]{2} + 0}_{\text{conv}} \qquad \boxed{= 3 + 3\sqrt[3]{2}}$$

* Comparison Test: Let f & g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

2) $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges

ex: $\int_1^\infty \frac{1}{x^3+1}\,dx \rightarrow$ compare to $\frac{1}{x^3}$ (if $\frac{1}{x^3}$ conv. then $\frac{1}{x^3+1}$ will also conv.)

$$0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty}\left(\frac{-1}{2}x^{-2}\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{-1}{2b^2} + \frac{1}{2}\right) = 0 + 1/2 = 1/2 \therefore \text{conv. so...} \int_1^\infty \frac{1}{x^3+1}\,dx$$

must also converge!