Series Packet

1979

a) $f(x) = \frac{1}{1-2x} \xrightarrow[\text{Series}]{\text{Maclaurin}} \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n$

$$T(x) = 1 + (2x) + (2x)^2 + (2x)^3 + \ldots + (2x)^n$$

$$\boxed{= 1 + 2x + 4x^2 + 8x^3 + \ldots + (2x)^n}$$

b) $|2x| < 1$

$-1 < 2x < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

check the end-pts

$x = -\frac{1}{2} \rightarrow \sum_{n=0}^{\infty} (2 \cdot -\frac{1}{2})^n = \sum_{n=0}^{\infty} (-1)^n \rightarrow$ diverges

$x = \frac{1}{2} \rightarrow \sum_{n=0}^{\infty} (2 \cdot \frac{1}{2})^n = \sum_{n=0}^{\infty} (1)^n \rightarrow$ diverges

$$\boxed{\therefore \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)}$$

c) $f(-\frac{1}{4}) = \frac{1}{1-2(-\frac{1}{4})} = \frac{1}{1+\frac{1}{2}}$

$= \frac{1}{3/2} = \frac{2}{3}$

error < 1% of $\frac{2}{3}$

error < $\frac{2}{300}$ or $.00\overline{6}$

next term

when $x = -\frac{1}{4} \rightarrow \sum_{n=0}^{\infty} (2 \cdot -\frac{1}{4})^n = \sum_{n=0}^{\infty} (-\frac{1}{2})^n$

$\left|\left(-\frac{1}{2}\right)^n\right| < \frac{2}{300}$ or $.00\overline{6}$

$\left(\frac{1}{2}\right)^n < \frac{2}{300}$ or $.00\overline{6}$

$\boxed{\text{when } n=8 \text{ the error is less than } 1\%}$

1980

$$A = \sum_{n=1}^{\infty} \frac{4n}{n^2+1}$$

a) Direct Comparison: $\frac{4n}{n^2+1} \approx \frac{4n}{n^2} \approx \frac{4}{n} > \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges from p-Series $p=1$

since $\frac{4n}{n^2+1}$ is greater than $\frac{1}{n}$, it must also diverge.

OR

LCT: $\lim_{n\to\infty} \left(\frac{\frac{4n}{n^2+1}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \left(\frac{4n}{n^2+1} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \frac{4n^2}{n^2+1} = 4$

so... since $1/n$ diverges so does $\frac{4n}{n^2+1}$.

b) $S = \sum_{n=1}^{\infty} \left(\frac{4n}{n^2+1} \cdot \frac{1}{2n} \right) = \boxed{\sum_{n=1}^{\infty} \frac{2}{n^2+1}}$

c) Direct Comp: $\frac{2}{n^2+1} < \frac{2}{n^2}$ p-Series: $p=2$ $\therefore$ converges

so... $\frac{2}{n^2+1}$ also converges.

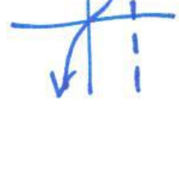
LCT: $\lim_{n\to\infty} \left(\frac{2}{n^2+1} \cdot \frac{n^2}{1} \right) = \lim_{n\to\infty} \frac{2n^2}{n^2+1} = 2$ $\therefore$ both converge.

Integral Test: $\lim_{b\to\infty} \int_1^b \frac{2}{x^2+1}\,dx = \lim_{b\to\infty} 2\tan^{-1}x \Big|_1^b$

$= \lim_{b\to\infty} 2\left[\tan^{-1}b - \tan^{-1}1\right]$

$= 2\left[\pi/2 - \pi/4\right] = 2(\pi/4)$

$= \pi/2$ $\therefore$ converge.



1981

$$S = \sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n \quad \text{where } t \neq 0.$$

a) when $t=1 \rightarrow S = \sum_{n=0}^{\infty} \left(\frac{1}{1+1}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ ← geo. series.

$$\text{Sum} = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

b) $\left|\frac{t}{1+t}\right| < 1$

$$-1 < \frac{t}{1+t} < 1$$

$$-1-t < t < 1+t$$

$$\boxed{\therefore t > -\tfrac{1}{2}}$$

$-1-t < t$

$-1 < 2t$

$-\frac{1}{2} < t$

$t < 1+t$

$0 < 1$

Always!

c) Sum > 10

$$\frac{1 \cdot \cancel{\frac{1+t}{1+t}}}{\cancel{\frac{1+t}{1+t}} \cdot 1 - \left(\frac{t}{1+t}\right)} > 10 \quad \text{* get a C.D.}$$

$$\frac{1+t}{(1+t)-t} > 10$$

$$\frac{1+t}{1} > 10$$

$$\boxed{t > 9}$$

1982

a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n}$

Maclaurin Series

$$\boxed{T(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n}}$$

b) $|x|<1$

$-1 < x < 1$

check the end-pts

$x=-1 \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} \rightarrow \underline{\underline{\text{div}}}$

$x=1 \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{(1)^n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Alt. Series
pos? ✓
decr? ✓
$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓
$\therefore$ con

$\boxed{\text{so}\ldots\ (-1, 1]}$

c) $\ln(3/2)$

$x = 1/2$

error $= |f(1/2) - T(1/2)|$

$= |.40546\ldots - .40729\ldots|$

$\boxed{= .00183}$

OR error = next term $(n=6)$

$= \frac{(1/2)^6}{6}$

$\boxed{= .00260}$

d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^{2n}}{2n}$ compare to $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$: plug in x^2
multiply by $1/2$

so... $\frac{1}{2} f(x^2) = \boxed{\frac{\ln(1+x^2)}{2}}$

1983

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{where } a_0 = 1 \; ; \; a_n = \left(\frac{7}{n}\right) a_{n-1} \text{ for } n \geq 1$$

a) $a_0 = 1$

$a_1 = \frac{7}{1}(1) = 7$

$a_2 = \frac{7}{2}(7) = \frac{49}{2}$

$a_3 = \frac{7}{3}\left(\frac{49}{2}\right) = \frac{343}{6}$

$$\boxed{T(x) = 1 + 7x + \frac{7^2x^2}{2!} + \frac{7^3x^3}{3!} + \dots + \frac{(7x)^n}{n!}}$$

b) RATIO:

$$\lim_{n\to\infty}\left[\frac{(7x)^{n+1}}{(n+1)!}\cdot\frac{n!}{(7x)^n}\right] = \lim_{n\to\infty}\frac{1}{n+1}\cdot 7x = 0(7x) = 0$$

$(7x)^n \cdot (7x)^1$

$(n+1)n!$

$$\boxed{\therefore \text{ interval of convergence } (-\infty, \infty)}$$

OR $\sum_{n=0}^{\infty} \frac{(7x)^n}{n!} = e^{7x} \rightarrow$ converges for all x's.

c) $f(x) = 1 + 7x + \frac{7^2x^2}{2!} + \frac{7^3x^3}{3!} + \dots + \frac{(7x)^n}{n!}$

$f'(x) = 7 + \frac{7^2 \cdot 2x}{2!} + \frac{7^3 \cdot 3x^2}{3!} + \dots + \frac{7^n \cdot nx^{n-1}}{n!}$

$= 7 + 7^2x + \frac{7^3x^2}{2!} + \frac{7^4x^3}{3!} + \dots + \frac{7^nx^{n-1}}{(n-1)!}$

$$f'(1) = 7\underbrace{\left(1 + 7 + \frac{7^2}{2!} + \frac{7^3}{3!} + \dots + \frac{7^{n-1}}{(n-1)!}\right)}_{e^7}$$

$$\boxed{f'(1) = 7e^7}$$

1984

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n \cdot n^n}{3^n \cdot n!} = \sum_{n=1}^{\infty} \frac{n^n}{n!}\left(\frac{x}{3}\right)^n$$

a) RATIO:

$$\lim_{n\to\infty}\left[\frac{\overbrace{(n+1)^{n+1}\left(\frac{x}{3}\right)^{n+1}}^{(n+1)^n\cdot\cancel{(n+1)}^1\cdot\cancel{\left(\frac{x}{3}\right)^n}\cdot\left(\frac{x}{3}\right)^1}}{\underbrace{(n+1)!}_{\cancel{(n+1)}\,\cancel{n!}}} \cdot \frac{\cancel{n!}}{n^n\cdot\cancel{\left(\frac{x}{3}\right)^n}}\right] = \underbrace{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n}_{e}\cdot\left(\frac{x}{3}\right) = e\cdot\frac{x}{3}$$

$$\left|e\cdot\frac{x}{3}\right| < 1$$

$$-1 < e\cdot\frac{x}{3} < 1$$

$$-\frac{3}{e} < x < \frac{3}{e}$$

$$\boxed{\text{Radius: } \frac{3}{e}}$$

b) $$f(-1) = \sum_{n=1}^{\infty}\frac{(-1)^n n^n}{3^n \cdot n!} = \frac{(-1)^1(1)^1}{3^1\cdot 1!} + \frac{(-1)^2(2)^2}{3^2\cdot 2!} + \frac{(-1)^3(3)^3}{3^3\cdot 3!} + \cdots$$

$$\boxed{f(-1) \approx -\frac{1}{3} + \frac{2}{9} - \frac{1}{6}}$$

c) error → <u>next</u> <u>term</u> $(n=4)$: $\frac{(-1)^4(4)^4}{3^4\cdot 4!} = \frac{256}{1944} = \boxed{\frac{32}{243}}$

1986

a) $f(x) = \sqrt{1+x} = (1+x)^{1/2}$ $\qquad f(0) = 1$

$f'(x) = 1/2(1+x)^{-1/2}$ $\qquad f'(0) = 1/2$

$f''(x) = -1/4(1+x)^{-3/2}$ $\qquad f''(0) = -1/4$

$f'''(x) = 3/8(1+x)^{-5/2}$ $\qquad f'''(0) = 3/8$

$$T(x) = 1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 + \cdots = \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots}$$

b) $g(x) = \sqrt{1+x^3}$ $\qquad T(x) = 1 + \frac{1}{2}(x^3) - \frac{1}{8}(x^3)^2 + \frac{1}{16}(x^3)^3 + \ldots$

$$\boxed{T(x) = 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16}}$$

c) $h'(x) = \sqrt{1+x^3}$ $\qquad \int h'(x) = \int \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16}\right) dx$

$h(0) = 4$ $\qquad h(x) = x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} + \ldots + C$

$$4 = h(0) = 0 + 0 - 0 + 0 + \ldots + C$$

$$C = 4$$

$$\boxed{h(x) = 4 + x + \frac{x^4}{8} - \frac{x^7}{56}}$$

1987

a) $f(x) = \frac{1}{1-2x} \xrightarrow[\text{Maclaurin Series}]{} \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n$

$T(x) = 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + \ldots + (2x)^n$

$\boxed{= 1 + 2x + 4x^2 + 8x^3 + 16x^4}$

b) $S = \frac{a_1}{1-r}$

$|2x| < 1$

$-1 < 2x < 1$

$-1/2 < x < 1/2$

check the end-pts

$x = -1/2 \rightarrow \sum_{n=0}^{\infty} (2 \cdot -1/2)^n = \sum_{n=0}^{\infty} (-1)^n \rightarrow$ div

$x = 1/2 \rightarrow \sum_{n=0}^{\infty} (2 \cdot 1/2)^n = \sum_{n=0}^{\infty} (1)^n \rightarrow$ div

$\boxed{\text{so} \ldots \ (-1/2, 1/2)}$

c) $g(x) = \frac{1}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$

$1 = A(1-x) + B(1-2x)$

$x = 1 \quad 1 = A(0) + B(-1) \quad B = -1$

$x = 1/2 \quad 1 = A(1/2) + B(0) \quad A = 2$

$g(x) = \frac{2}{1-2x} + \frac{-1}{1-x}$

$= 2\left(\frac{1}{1-2x}\right) + -1\left(\frac{1}{1-x}\right)$

$= 2(1 + 2x + 4x^2 + 8x^3 + 16x^4) - (1 + x + x^2 + x^3 + x^4)$

$= 2 + 4x + 8x^2 + 16x^3 + 32x^4 - 1 - x - x^2 - x^3 - x^4$

$\boxed{= 1 + 3x + 7x^2 + 15x^3 + 31x^4}$

1988

$$\sum_{k=0}^{\infty} \frac{2^k \cdot x^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1}{\ln(k+2)} \cdot (2x)^k$$

RATIO:

$$\lim_{n\to\infty} \left[\frac{\overset{(2x)^n \cdot (2x)^1}{(2x)^{n+1}}}{\ln(n+1+2)} \cdot \frac{\ln(n+2)}{\cancel{(2x)^n}} \right] = \underbrace{\lim_{n\to\infty} \frac{\ln(n+2)}{\ln(n+3)}}_{\to 1} \cdot 2x = 2x$$

$$|2x| < 1$$

$$-1 < 2x < 1$$

$$-\tfrac{1}{2} < x < \tfrac{1}{2}$$

Check the end-pts

$$x = -\tfrac{1}{2} \to \sum_{k=0}^{\infty} \frac{(2 \cdot -\frac{1}{2})^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$$

Alt. Series

pos? ✓

decr? ✓

$\lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0$ ✓

$\therefore$ conv

$$x = \tfrac{1}{2} \to \sum_{k=0}^{\infty} \frac{(2 \cdot \frac{1}{2})^k}{\ln(k+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(+1)^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$$

$$\boxed{\text{so} \ldots \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)}$$

Direct Compar: $\frac{1}{\ln(k+2)} > \frac{1}{k+2} \leftarrow$ diverges

so ↑ must also diverge

1990

$f(x) = \frac{1}{x-1}$

a) Taylor about $x=2$ $\quad \frac{1}{x-1} = \frac{1}{1+(x-2)} = \boxed{1-(x-2)+(x-2)^2-(x-2)^3+\ldots+(-1)^n(x-2)^n}$

b) $\ln|x-1|$ relates to $\frac{1}{x-1}$ by integration

$$\int g(x) = \int \frac{1}{x-1}\,dx$$

$$G(x) = \int\left[1-(x-2)+(x-2)^2-(x-2)^3+\ldots+(-1)^n(x-2)^n\right]dx$$

$$\ln|x-1| = x-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\frac{(x-2)^4}{4}+\ldots+\frac{(-1)^n(x-2)^{n+1}}{n+1}+C$$

about $x=2$ $\rightarrow \ln|2-1| = 2-0+0-0+\ldots+C$

$$0 = 2+C$$

$$C=-2$$

$$\boxed{\ln|x-1| = -2+x-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\frac{(x-2)^4}{4}+\ldots+\frac{(-1)^n(x-2)^{n+1}}{n+1}}$$

c) $\ln(3/2) = .40546\ldots$ $\quad$ so $\quad .35546\ldots < x < .45546\ldots$

$x = 5/2$

$-2+5/2 = 1/2$ No

$-2+5/2-1/8 = \boxed{3/8 \text{ or } .375}$

* add up the partial sums.