1991

$$f(t)=\frac{4}{1+t^2}\ \xrightarrow[\text{Series}]{\text{Maclaurin}}\ \frac{1}{1+x}=1-x+x^2-x^3+x^4-\ldots+(-1)^n x^n$$

a) * multiply by 4
plug in t^2

$$T(x)=\left[1-(t^2)+(t^2)^2-(t^2)^3+\ldots+(-1)^n(t^2)^n\right]\cdot 4$$

$$\boxed{T(x)=4-4t^2+4t^4-4t^6+\ldots+(-1)^n 4t^{2n}}$$

b)
$$G(x)=\int_0^x f(t)\,dt=\int_0^x\left(4-4t^2+4t-4t^6+\ldots\right)dt$$

$$=4t-\frac{4t^3}{3}+\frac{4t^5}{5}-\frac{4t^7}{7}+\ldots+\frac{4t^{2n+1}}{2n+1}(-1)^n\Big|_0^x$$

$$\boxed{=4x-\frac{4x^3}{3}+\frac{4x^5}{5}-\frac{4x^7}{7}+\ldots+\frac{4x^{2n+1}}{2n+1}(-1)^n}$$

c) RATIO:

$$\lim_{n\to\infty}\left[\frac{\cancel{4}x^{2(n+1)+1}}{2(n+1)+1}\cdot\frac{2n+1}{\cancel{4}x^{2n+1}}\right]=\lim_{n\to\infty}\left[\frac{2n+1}{2n+3}\cdot\frac{\overset{\cancel{x^{2n}}\cdot x^{\cancel{3}2}}{x^{2n+3}}}{\underset{\cancel{x^{2n}}\cdot\cancel{x^1}}{x^{2n+1}}}\right]=\cancel{\lim_{n\to\infty}}\overset{1}{\frac{\cancel{2n+1}}{2n+3}}\cdot x^2$$

$$=x^2$$

$$|x^2|<1$$

$$\cancel{\sqrt{-1}}<\sqrt{x^2}<\sqrt{1}$$

$x<1$ or $x>-1$

$-1<x<1$

check the end-pts

$$x=-1\to\sum_{n=0}^{\infty}\frac{(-1)^n(4)(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{4}{2n+1}(-1)^{3n+1}\to(-1)^{n+1}$$

Alt. Series
pos? ✓
decr? ✓
$\lim_{n\to\infty}\frac{4}{2n+1}=0$ ✓
∴ conv.

$$x=1\to\sum_{n=0}^{\infty}\frac{(-1)^n 4\cancel{(1)^{2n+1}}}{2n+1}$$

$$=\sum_{n=0}^{\infty}(-1)\cdot\frac{4}{2n+1}$$

∴ conv.

$$\boxed{\text{So}\ldots\ [-1,1]}$$

1992

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$$ for $p \geq 0$

a) $p > 1$

$$\frac{1}{n^p \ln(n)} < \frac{1}{n^p}$$ for $n \geq 3$ (Direct Comparison)

$\frac{1}{n^p}$ converges if $p > 1$ using the p-Series test.

so... if $\frac{1}{n^p \ln(n)}$ is $< \frac{1}{n^p}$, then it must also converge.

b) $p = 1$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \longrightarrow$$ (Integral Test)

$$\int_2^{\infty} \frac{1}{x \ln(x)}\,dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln(x)}\,dx$$

$u = \ln x$
$du = \frac{1}{x}dx$

$$= \lim_{b \to \infty} \int_2^b \frac{1}{u}\,du$$

$$= \lim_{b \to \infty} \ln(\ln x) \Big|_2^b$$

$$= \lim_{b \to \infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty$$

$\therefore$ the series diverges.

c) $0 \leq p < 1 \longrightarrow$ (Direct Comparison)

$$\frac{1}{n^p \ln(n)} \geq \frac{1}{n \ln(n)}$$ if p is between 0 & 1.

Just proved this diverged in part b.

since $\frac{1}{n^p \ln(n)} \geq \frac{1}{n \ln(n)}$ then it must also diverge.

1993

$f(x) = e^{x/2}$

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$e^{x/2} = 1 + (x/2) + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \ldots + \frac{(x/2)^n}{n!}$

$$\boxed{= 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \ldots + \frac{x^n}{2^n \cdot n!}}$$

b) $g(x) = \frac{e^{x/2} - 1}{x} = \frac{\left[\cancel{1} + x/2 + x^2/2^2 \cdot 2! + x^3/2^3 \cdot 3! + \ldots\right] - \cancel{1}}{x}$

$$\boxed{= \frac{1}{2} + \frac{x}{2^2 \cdot 2!} + \frac{x^2}{2^3 \cdot 3!} + \frac{x^3}{2^4 \cdot 4!} + \ldots + \frac{x^{n-1}}{2^n \cdot n!}}$$

c) $g'(2) = ?$ $\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$

$g(x) = \frac{e^{x/2} - 1}{x}$

$g'(x) = \frac{x(\frac{1}{2}e^{x/2}) - (e^{x/2} - 1)(1)}{x^2}$

$g'(2) = \frac{\cancel{2}(\cancel{\frac{1}{2}}e^{2/2}) - (e^{2/2} - 1)}{2^2} = \frac{e^1 - e^1 + 1}{4}$

$$\boxed{= \frac{1}{4}}$$

when $x = 2$

$\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n \cdot n!}$

$\frac{d}{dx}\left(\frac{x^{n-1}}{2^n \cdot n!}\right) = \frac{1}{2^n \cdot n!}(n-1)x^{n-2}$

$\cancel{2^n} \cdot 2^{-2}$

$@ x=2 \longrightarrow \frac{1}{\cancel{2^n} \cdot n!}(n-1)(2)^{n-2}$

$= \frac{n-1}{4n!}$

Does $\frac{n-1}{4n!} = \frac{n}{4(n+1)!}$?

Yes, Both are $\frac{\text{term}}{4(\text{Next term})!}$ ✓

1994

$f(x) = e^{-2x^2}$ $\qquad$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

a) $e^{-2x^2} = 1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \ldots + \frac{(-2x^2)^n}{n!}$

$$\boxed{= 1 - 2x^2 + \frac{4x^4}{2!} - \frac{8x^6}{3!} + \ldots + \frac{(-2)^n x^{2n}}{n!} \quad \underline{\underline{\text{OR}}} \quad \frac{(-1)^n(2)^n x^{2n}}{n!}}$$

b) since e^x converges for $-\infty < x < \infty$, then

e^{-2x^2} converges for $-\infty < -2x^2 < \infty \rightarrow \boxed{-\infty < x < \infty}$

$\underline{\underline{\text{OR}}}$ RATIO:

$$\lim_{n\to\infty}\left[\frac{\overset{\cancel{(-2)^n}\cdot(-2)^1\,\cancel{x^{2n}}\cdot x^2}{(-2)^{n+1}x^{2(n+1)}}}{\underset{(n+1)\cancel{n!}}{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{(-2)^n}\,\cancel{x^{2n}}}\right] = \cancel{\lim_{n\to\infty}\frac{1}{n+1}}^{\,0} \cdot x^2 = 0 \cdot x^2 = 0 < 1 \text{ Always!}$$

$$\boxed{\therefore -\infty < x < \infty}$$

c) $f(x) = e^{-2x^2}$ $\qquad$ $f(.6) = .4867\ldots$

$g(x) = 1 - 2x^2 + \frac{4x^4}{2!} - \frac{8x^6}{3!}$ $\qquad$ $g(.6) = .476992$

$|f(.6) - g(.6)| < .02$

let $x = |.6|$

$.00976 < .02$ ✓

$\underline{\underline{\text{OR}}}$ error $< .02$

next term: $\frac{16x^8}{4!} \rightarrow \frac{16(0.6)^8}{4!} = .0112 < .02$ ✓

<u>1996</u>

$$f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots + \frac{x^n}{(n+1)!}$$

a) $f'(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \dots + \frac{n\,x^{n-1}}{(n+1)!}$ $\boxed{f'(0) = \frac{1}{2}}$

coefficients: $\frac{1}{(n+1)!}$

each derv: $n!\,\frac{1}{(n+1)!}$ so... $f^{17}(0) = 17!\left(\frac{1}{18!}\right) = \frac{\cancel{17!}}{18\cdot\cancel{17!}} = \boxed{\frac{1}{18}}$

b) RATIO:

$$\lim_{n\to\infty}\left[\frac{\overset{\cancel{x^n}\cdot x^1}{x^{n+1}}}{\underset{(n+2)\cancel{(n+1)!}}{(n+1+1)!}}\cdot\frac{\cancel{(n+1)!}}{\cancel{x^n}}\right] = \lim_{n\to\infty}\frac{1}{n+2}^{\nearrow 0}\cdot x = 0\cdot x = 0$$

$\boxed{\therefore -\infty < x < \infty}$

c) $g(x) = x\,f(x) = x\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots + \frac{x^n}{(n+1)!}\right]$

$$\boxed{= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n+1}}{(n+1)!}}$$

d) $g(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n+1}}{(n+1)!} = e^x - 1$

$$x\cdot f(x) = e^x - 1$$

$$\boxed{f(x) = \begin{cases} \frac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}}$$

1998

$f(0)=5$

$f'(0)=-3$

$f''(0)=1$

$f'''(0)=4$

a) $T(x)=5-3x+\frac{1x^2}{2!}+\frac{4x^3}{3!}$

$$\boxed{=5-3x+\frac{x^2}{2}+\frac{2x^3}{3}}$$

$$\boxed{T(.2)=4.425}$$

b) $g(x)=f(x^2)=5-3(x^2)+\frac{(x^2)^2}{2}+\frac{2(x^2)^3}{3}$

$=5-3x^2+\frac{x^4}{2}+\cancel{\frac{2x^6}{3}}$ → Fourth Degree Poly

$$\boxed{=5-3x^2+\frac{x^4}{2}}$$

c) $h(x)=\int_0^x f(t)\,dt=\int_0^x 5-3t+\frac{t^2}{2}+\frac{2t^3}{3}\Big)\,dt$

$=5t-\frac{3t^2}{2}+\frac{t^3}{6}+\frac{2t^4}{12}\Big|_0^x$

$=5x-\frac{3x^2}{2}+\frac{x^3}{6}+\cancel{\frac{x^4}{6}}$ → Third Degree Poly

$$\boxed{=5x-\frac{3x^2}{2}+\frac{x^3}{6}}$$

d) $f(1)=3$ $\quad h(1)=\int_0^1 f(t)\,dt$ but the exact value can not be determined because $f(t)$ is only known @ $t=0$ & $t=1$.

1999

$f(2) = -3$

$f'(2) = 5$

$f''(2) = 3$

$f'''(2) = -8$

a) $T(x) = -3 + 5(x-2) + \frac{3(x-2)^2}{2!} - \frac{8(x-2)^3}{3!}$

$$\boxed{= -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{4}{3}(x-2)^3}$$

$$\boxed{T(1.5) = -4.958}$$

b) $\left|f^4(x)\right| \le 3$ using $x=2$ $f^4(2) \le 3$ or $f^4(2) = 3$

error → next term: $\frac{3(x-2)^4}{4!} \to \frac{3(1.5-2)^4}{4!}$

$$\overset{\text{next term}}{\underset{}{\text{error}}} = \left|\overset{f(1.5)}{\text{actual}} - \overset{T(1.5)}{\text{est}}\right|$$

$$\boxed{\text{error} = .0078125}$$

error + est = actual

$\pm .0078125 + (-4.958) = \text{actual}$

$-4.966 > -5$ $\boxed{\therefore f(1.5) \ne -5}$

c) $P(x) \to g(x) = f(x^2+2) = -3 + 5(x^2+2-2) + \frac{3}{2}(x^2+2-2)^2$

$$\boxed{P(x) = \underset{a_0}{\underset{\uparrow}{-3}} + \underset{a_2}{\underset{\uparrow}{5}}x^2 + \underset{a_4}{\underset{\uparrow}{\tfrac{3}{2}}}x^4}$$

$P(0) = 3$ (a_0)

$P'(0) = 0$ (a_1)

$P''(0) = 10$ (a_2) →

$P'''(0) = 0$ (a_3)

⋮

$g(x) = 3 + 0x + \frac{a_2 x^2}{2!} + a_3 x^3$

$\frac{a_2}{2!} = 5$

$a_2 = 10$

$P'(0) = 0 \therefore x=0$ is a crt. pt.
$P''(0) = 10$ then $P(x)$ is cc up
$\therefore x=0$ must be a minimum

2001

$$f(x) = \frac{1}{3} + \frac{2x}{3^2} + \frac{3x^2}{3^3} + \dots + \frac{(n+1)x^n}{3^{n+1}}$$

a) RATIO.

$$\lim_{n\to\infty}\left[\frac{(n+2)\overset{\cancel{x^n}\cdot x^1}{x^{n+1}}}{\underset{\cancel{3^n}\cdot 3^2}{3^{n+2}}}\cdot\frac{\overset{\cancel{3^n}\cdot\cancel{3^1}}{3^{n+1}}}{(n+1)\cancel{x^n}}\right] = \lim_{n\to\infty}\cancel{\frac{n+2}{n+1}}^{\,1}\cdot\frac{x}{3} = \frac{x}{3}$$

$$\left|\frac{x}{3}\right| < 1$$

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

check the end-pts

$$x=-3 \rightarrow \sum_{n=0}^{\infty}\frac{(n+1)(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{(n+1)(-1)^n\cancel{(3)^n}}{\underset{\cancel{3^n}\cdot 3^1}{3^{n+1}}} = \sum_{n=0}\frac{n+1}{3}(-1)^n$$

Alt. Series
pos? ✓
decr? NO ∴ div.

$$x=3 \rightarrow \sum_{n=0}^{\infty}\frac{(n+1)\cancel{(3)^n}}{\underset{\cancel{3^n}\cdot 3^1}{3^{n+1}}} = \sum_{n=0}^{\infty}\frac{n+1}{3} \rightarrow \text{div.}$$

$$\boxed{\text{so}\dots\ (-3,3)}$$

b)

$$\lim_{x\to 0}\frac{f(x)-1/3}{x} = \lim_{x\to 0}\left(\frac{2}{3^2}+\frac{3x}{3^3}+\frac{4x^2}{3^4}+\dots\right) = \frac{2}{3^2} = \boxed{\frac{2}{9}}$$

$$\frac{f(x)-1/3}{x} = \frac{\left(\cancel{1/3}+\frac{2x}{3^2}+\frac{3x^2}{3^3}+\dots\right)-\cancel{1/3}}{x} = \frac{2}{3^2}+\frac{3x}{3^3}+\frac{4x^2}{3^4}+\dots$$

c)

$$\int_0^1 f(x)\,dx = \int_0^1\left(\frac{1}{3}+\frac{2x}{3^2}+\frac{3x^2}{3^3}+\dots\right)dx$$

$$= \left[\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\dots+\frac{x^{n+1}}{3^{n+1}}\right]\Bigg|_0^1$$

$$\boxed{= \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3}+\dots\frac{1}{3^{n+1}}}$$

d)

$$\text{Sum} = \frac{a_1}{1-r} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \boxed{\frac{1}{2}}$$

2002

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \ldots + \frac{(2x)^{n+1}}{n+1}$$

a) RATIO:

$$\lim_{n\to\infty}\left[\frac{\overset{\cancel{(2x)^n}\cdot(2x)^{\overset{1}{\cancel{2}}}}{(2x)^{n+2}}}{n+2}\cdot\frac{n+1}{\underset{\cancel{(2x)^n}\cdot\cancel{(2x)^1}}{(2x)^{n+1}}}\right] = \overset{1}{\cancel{\lim_{n\to\infty}\frac{n+1}{n+2}}}\cdot 2x = 2x$$

$|2x| < 1$

$-1 < 2x < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

check the endpts:

$$x = -\tfrac{1}{2} \rightarrow \sum_{n=0}^{\infty} \frac{(2\cdot -\frac{1}{2})^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

Alt. Series

pos? ✓

decr? ✓

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ ✓

$\therefore$ conv.

$$x = \tfrac{1}{2} \rightarrow \sum_{n=0}^{\infty} \frac{(2\cdot\frac{1}{2})^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(1)^{n+1}}{n+1}$$

Direct Comp: $\frac{1}{n+1} < \frac{1}{n}$ ← diverges

then ↑ must also diverge.

$$\boxed{\text{So}\ldots \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)}$$

b) $f(x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \ldots$

$$f'(x) = 2 + \frac{8x}{2} + \frac{24x^2}{3} + \frac{64x^3}{4} + \ldots + \frac{\cancel{(n+1)}(2x)^n(2)^1}{\cancel{(n+1)}}$$

$$\boxed{= 2 + 4x + 8x^2 + 16x^3 + \ldots + (2)^{n+1}x^n}$$

c)

$$f'(-\tfrac{1}{3}) = 2 + 4(-\tfrac{1}{3}) + 8(-\tfrac{1}{3})^2 + 16(-\tfrac{1}{3})^3 + \ldots + 2^{n+1}(-\tfrac{1}{3})^n$$

$$= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots + (-1)^n\left(\frac{2}{3}\right)^n\cdot 2 \text{ OR } \left(\frac{-2}{3}\right)^n\cdot 2$$

$$\text{Sum} = \frac{a_1}{1-r} = \frac{2}{1-(-\frac{2}{3})} = \frac{2}{\frac{5}{3}} = \boxed{\frac{6}{5}}$$

2002 Form B

$\ln\left(\frac{1}{1-x}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}$ w/ interval of conv: $[-1,1)$

a) $\ln\left(\frac{1}{1+3x}\right)=\sum_{n=1}^{\infty}\frac{(-3x)^n}{n}=\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n(3x)^n}{n}}$

$-1\le -3x<1$

$\frac{1}{3}\ge x>\frac{-1}{3}$ OR $\boxed{\frac{-1}{3}<x\le\frac{1}{3}}$

b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=\ln\left[\frac{1}{1-(-1)}\right]=\boxed{\ln\left(\frac{1}{2}\right)}$

c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}\to$ converge

but... $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\to$ diverge

p-Series: $2p\le 1$ to div

$p\le 1/2$

Alt. Series

$\frac{1}{n^p}\ge 0\to$ all p

$\frac{1}{n^p}\ge\frac{1}{(n+1)^p}\to p>0$

$\lim_{n\to\infty}\frac{1}{n^p}=0\to p>0$

$\boxed{\text{so...}\ 0<p\le 1/2}$

d) $\sum_{n=1}^{\infty}\frac{1}{n^p}\to$ diverge

$p\le 1$ to div: p-Series

but... $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\to$ converge

p-Series: $2p>1$ to conv.

$p>1/2$

$\boxed{\text{so...}\ 1/2<p\le 1}$