2003

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots+\frac{(-1)^n x^{2n}}{(2n+1)!}$$

$f(0)$ $\quad \frac{f''(0)x^2}{2!}$ $\quad \frac{f^4(0)x^4}{4!}$ ...

a) $\boxed{\begin{array}{l} f'(0)=0 \rightarrow \text{crt. pt @ } x=0 \\ f''(0)=-1/3 \rightarrow \text{cc down} \therefore \\ \qquad x=0 \text{ is a maximum}\end{array}}$

$$-\frac{1}{3!}=\frac{f''(0)}{2!}$$

$$-\frac{\cancel{2!}}{\underset{3\cdot\cancel{2!}}{3!}}=f''(0)$$

b) $\underbrace{1-\frac{1}{3!} \text{ approx. } f(1)}_{\text{next term: } \frac{x^4}{5!}}$ show error $<\frac{1}{100}$

@ $x=1$: $\frac{(1)^4}{5!}<\frac{1}{100}$

$\frac{1}{120}<\frac{1}{100}$ ✓

c) $x\cdot y'+\overset{f(x)}{y}=\cos x$

$\overset{f'(x)}{y'}=-\frac{2}{3!}x+\frac{4x^3}{5!}-\frac{6x^5}{7!}+\dots+\frac{(-1)^n\cdot 2n x^{2n-1}}{(2n+1)!}$

$$x\left[-\frac{2x}{3!}+\frac{4x^3}{5!}-\frac{6x^5}{7!}+\dots+\frac{(-1)^n\cdot 2n x^{2n-1}}{(2n+1)!}\right]+\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots+\frac{(-1)^n x^{2n}}{(2n+1)!}\right]=\cos x$$

$$-\frac{2x^2}{3!}+\frac{4x^4}{5!}-\frac{6x^6}{7!}+\dots+\frac{(-1)^n\cdot 2n x^{2n}}{(2n+1)!}+1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots+\frac{(-1)^n x^{2n}}{(2n+1)!}=\cos x$$

$$1-\frac{3x^2}{3!}+\frac{5x^4}{5!}-\frac{7x^6}{7!}+\dots+\frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!}=\cos x$$

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots+\frac{(-1)^n x^{2n}}{2n!}=\cos x \checkmark$$

2003 Form B

$$f^{n}(2) = \frac{(n+1)!}{3^n}$$

a) $f(2) = 1$

$f'(2) = \frac{2!}{3^1} = \frac{2}{3}$

$f''(2) = \frac{3!}{3^2}$

$f'''(2) = \frac{4!}{3^3}$

$$T(x) = 1 + \frac{2}{3}(x-2) + \frac{\frac{\overset{3\cdot\cancel{2!}}{3!}}{3^2}(x-2)^2}{\cancel{2!}} + \frac{\frac{\overset{4\cdot\cancel{3!}}{4!}}{3^3}(x-2)^3}{\cancel{3!}} + \dots$$

$$\boxed{= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \dots + \frac{(n+1)(x-2)^n}{3^n}}$$

b) RATIO: $\lim_{n\to\infty}\left[\frac{(n+2)\overset{(x-2)^n\cdot(x-2)}{(x-2)^{n+1}}}{\underset{3^n\cdot 3^1}{3^{n+1}}} \cdot \frac{\cancel{3^n}}{(n+1)\cancel{(x-2)^n}}\right] = \lim_{n\to\infty} \overset{1}{\frac{n+2}{n+1}} \cdot \frac{(x-2)}{3} = \frac{x-2}{3}$

$\left|\frac{x-2}{3}\right| < 1$

$-1 < \frac{x-2}{3} < 1$

$-3 < x-2 < 3$ $\quad \therefore$ Radius $= 3$

$-1 < x < 5$

c) $g(2) = 3$

$g'(2) = f(2) = 1$

$g''(2) = f'(2) = 2/3$

$g'''(2) = f''(2) = 3!/3^2$

$$g(x) = 3 + (x-2) + \frac{2/3\,(x-2)^2}{\cancel{2}!} + \frac{\cancel{3!}/3^2\,(x-2)^3}{\cancel{3!}} + \dots$$

$$\boxed{= 3 + (x-2) + \frac{(x-2)^2}{3} + \frac{(x-2)^3}{3^2} + \dots + \frac{(x-2)^n}{3^{n-1}}}$$

d) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n-1}} = \sum_{n=0}^{\infty} 3\left(\frac{x-2}{3}\right)^n \rightarrow$ geo. series

$\left|\frac{x-2}{3}\right| < 1$

$\downarrow$ *done in part b.

$-1 < x < 5$

$\boxed{\therefore\ x = -2 \text{ is NOT in the interval of convergence}}$

2004

$f(x) = \sin\left(5x + \pi/4\right)$ * Not centered @ x=0 ∴ can't use Maclaurin series!

a) $f(0) = \sin(\pi/4) = \sqrt{2}/2$

$f'(x) = 5\cos\left(5x + \pi/4\right) \qquad f'(0) = \frac{5\cdot\sqrt{2}}{2}$

$f''(x) = -25\cos\left(5x + \pi/4\right) \qquad f''(0) = \frac{-25\sqrt{2}}{2}$

$f'''(x) = -125\cos\left(5x + \pi/4\right) \qquad f'''(0) = \frac{-125\sqrt{2}}{2}$

$$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{\frac{25\sqrt{2}}{2}}{2!}x^2 - \frac{\frac{125\sqrt{2}}{2}}{3!}x^3$$

$$\boxed{= \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{4}x^2 - \frac{125\sqrt{2}}{12}x^3}$$

OR

$$\boxed{\frac{\sqrt{2}}{2}\left[1 + 5x - \frac{5^2x^2}{2!} - \frac{5^3x^3}{3!}\right]}$$

b) $x^{22} \rightarrow \boxed{-\frac{5^{22}\cdot\sqrt{2}}{22!\ 2}}$

c) $\text{error} = \left| f(1/10) - P(1/10)\right| < \frac{1}{100}$

$\left|.9595\ldots - .9575\ldots\right| < \frac{1}{100}$

$.0020091971 < .01$ ✓

OR error = next term: $\frac{5^4x^4}{4!}$

$\frac{5^4(1/10)^4}{4!} < \frac{1}{100}$

$.002604\ldots < \frac{1}{100}$

$\frac{1}{384} < \frac{1}{100}$ ✓

d) $\int_0^x f(t)\,dt = G(x)$

$G(x) = \int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right)dt$

$= \frac{\sqrt{2}}{2}t + \frac{5\sqrt{2}}{4}t^2 - \frac{25\sqrt{2}}{12}t^3 \Big|_0^x$

$$\boxed{= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3}$$

2004 Form B

$$T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$$

$f(2)$ $\quad \frac{f''(2)}{2!}$ $\quad \frac{f'''(2)}{3!}$

a) $f(2) = 7$

$f''(2) = -18$

$$-9 = \frac{f''(2)}{2!}$$

$$-9 \cdot 2! = f''(2)$$

b) $f'(2) = 0$ so $x=0$ is a crt. pt.

$f''(2) = -18$ cc down $\boxed{\therefore\ x=0 \text{ is a maximum}}$

c) $T(0) = 7 - 9(0-2)^2 - 3(0-2)^3$

$= -5$ so... $\boxed{f(0) \approx -5}$

$\boxed{\text{Not enough info. The approx. is about } x=2 \text{ not } x=0.}$

d) $|f^4(x)| \leq 6$ on $[0,2]$ so... $f^4(2) = 6$ error = next term

$$\frac{6(x-2)^4}{4!}$$

$$\text{error} = \frac{6(x-2)^4}{4!} = 4$$

error $= |\text{act} - \text{est}|$

$4 = |\text{act} - (-5)|$

$4 = |\text{act} + 5|$ $\boxed{\therefore \text{ actual value of } f(0) \text{ must be } (-)}$

2005

$f(2)=7$ every odd derv $=0$ even derv: $f^{n}(2)=\frac{(n-1)!}{3^n}$

a) $f(2)=7$

$f''(2)=1/3^2$

$f^4(2)=3!/3^4$

$f^6(2)=5!/3^6$

$$T(x)=7+\frac{1/3^2(x-2)^2}{2!}+\frac{\cancel{3!}/3^4(x-2)^4}{4!\ 4\cdot\cancel{3!}}+\frac{\cancel{5!}/3^6(x-2)^6}{6!\ 6\cdot\cancel{5!}}$$

$$\boxed{=7+\frac{(x-2)^2}{3^2\cdot 2}+\frac{(x-2)^4}{3^4\cdot 4}+\frac{(x-2)^6}{3^6\cdot 6}}$$

b) coeff of $(x-2)^{2n}$ ➡ $\boxed{\frac{1}{3^{2n}(2n)}}$

c) $7+\sum_{n=1}^{\infty}\frac{(x-2)^{2n}}{3^{2n}(2n)}$

RATIO: $$\lim_{n\to\infty}\left[\frac{\overset{\cancel{(x-2)^{2n}}(x-2)^2}{(x-2)^{2(n+1)}}}{\underset{\cancel{3^{2n}}\cdot 3^2}{3^{2(n+1)}}\underset{2n+2}{(2(n+1))}}\cdot\frac{\cancel{3^{2n}}(2n)}{\cancel{(x-2)^{2n}}}\right]$$

$$=\cancel{\lim_{n\to\infty}\overset{\nearrow 1}{\frac{2n}{2n+2}}}\cdot\frac{(x-2)^2}{3^2}=\frac{(x-2)^2}{9}$$

$$\left|\frac{(x-2)^2}{9}\right|<1$$

$$-1<\frac{(x-2)^2}{9}<1$$

$$-9<(x-2)^2<9$$

$x-2<3$ or $x-2>-3$

$x<5$ $\quad$ $x>-1$

$$\boxed{-1<x<5}$$

check the end-pts:

$x=-1\rightarrow\sum_{n=1}^{\infty}\frac{(-3)^{2n}}{3^{2n}(2n)}$

$=\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2n}$

$=\sum_{n=1}^{\infty}\frac{1}{2n}\rightarrow$ div

$x=5$ $\quad\sum_{n=1}^{\infty}\frac{(3)^{2n}}{3^{2n}\cdot 2n}=\sum_{n=1}^{\infty}\frac{1}{2n}$

div

$$\boxed{\therefore(-1,5)}$$

2005 Form B

$$f^n(0) = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2} \text{ for } n \geq 2 \text{ ; horz tangent @ } x=0 \text{ ; } f(0)=6$$

a) $f'(0) = 0 \rightarrow$ horz. tangent $\therefore x=0$ is crt. pt.

$$n=2: \; f''(0) = \frac{(-1)^3(3)!}{5^2(1)^2} = \frac{-6}{25} \; \therefore \text{ cc down so} \ldots \; \boxed{x=0 \text{ is a maximum}}$$

b) $$f'''(0) = \frac{(-1)^4(4)!}{5^3(2)^2} = \frac{24}{125(4)} = \frac{6}{125}$$

$$T(x) = 6 - \frac{\frac{6}{25}}{2!}x^2 + \frac{\frac{6}{125}}{3!}x^3 = \boxed{6 - \frac{3}{25}x^2 + \frac{1}{125}x^3}$$

c) each term: $$\frac{f^n(0)x^n}{n!} = \frac{(-1)^{n+1}\overbrace{(n+1)!}^{(n+1)\cancel{n!}}\,x^n}{5^n(n-1)^2\cancel{n!}} = \frac{(-1)^{n+1}(n+1)x^n}{5^n(n-1)^2}$$

RATIO:
$$\lim_{n\to\infty}\left[\frac{(n+2)\overbrace{x^{n+1}}^{\cancel{x^n}\cdot x^1}}{\underbrace{5^{n+1}}_{\cancel{5^n}\cdot 5}(n)^2}\cdot\frac{\cancel{5^n}(n-1)^2}{(n+1)\cancel{x^n}}\right] = \lim_{n\to\infty}\frac{(n+2)(n-1)^2}{5n^2(n+1)}\cdot x = \frac{1}{5}\cdot x = \frac{x}{5}$$

$$\left|\frac{x}{5}\right| < 1$$

$$-1 < \frac{x}{5} < 1$$

$$-5 < x < 5 \quad \boxed{\therefore \text{Radius} = 5}$$

2006

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \ldots + \frac{(-1)^n x^n \cdot n}{n+1}$$

$$g(x) = 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x}{6!} + \ldots + \frac{(-1)^n x^n}{(2n)!}$$

a) RATIO:

$$\lim_{n\to\infty}\left[\frac{\cancel{x^n}\cdot x^1 (n+1)}{n+2}\cdot\frac{n+1}{\cancel{x^n}\cdot n}\right] = \cancel{\lim_{x\to\infty}\frac{(n+1)^2}{n(n+2)}}^{\,1}\cdot x = x \qquad |x|<1$$

$$-1 < x < 1$$

check end-pts

$$x=-1 \rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{2n} n}{n+1} = \sum_{h=1}^{\infty}\frac{n}{n+1}$$

n^{th} Term Test

$\lim_{n\to\infty} n/n+1 = 1 \therefore \underline{dN}$

$$x=1 \rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n(1)^n\cdot n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n n}{n+1}$$

Alt. series

pos? ✓

decr? NO $\therefore$ dN

$$\boxed{\text{so}\ldots\ (-1,1)}$$

b) $y = f(x) - g(x)$ passes thru $(0,-1)$

$$y' = f'(x) - g'(x) = \left[\frac{-1}{2} + \frac{4x}{3} - \frac{9x^2}{4} + \ldots\right] - \left[\frac{-1}{2} + \frac{2x}{4!} - \frac{3x^2}{6!}\ldots\right]$$

$$\boxed{y'(0) = f'(0) - g'(0) = -\frac{1}{2} + \left(\frac{+1}{2}\right) = 0}$$

$$y'' = f''(x) - g''(x) = \left[4/3 - \frac{18x}{4} + \ldots\right] - \left[\frac{1}{12} - \frac{6x}{6!} + \ldots\right]$$

$$\boxed{y''(0) = f''(0) - g''(0) = \frac{4}{3} - \frac{1}{12} = \frac{15}{12}}$$

$$\boxed{\begin{array}{c}\text{since } y'(0)=0 \ \& \ y''(0)>0 \text{ (cc up)}\\ x=0 \text{ is a minimum}\end{array}}$$

2007

$f(x) = e^{-x^2}$ $\qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

a) $e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots + \frac{(-x^2)^n}{n!}$

$$\boxed{= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!}}$$

b) $\lim_{x \to 0} \frac{1 - x^2 - f(x)}{x^4}$

$\left[\cancel{1} - \cancel{x^2} + \left[\cancel{-1} + \cancel{x^2} - \frac{x^4}{2!} + \frac{x^6}{3!} + \dots\right]\right] / x^4$

$= \left(\frac{-x^4}{2!} + \frac{x^6}{3!} - \frac{x^8}{4!} + \dots\right) / x^4$

$= -\frac{1}{2} + \frac{x^2}{3!} - \frac{x^4}{4!} + \dots$

$\lim_{x \to 0} \left(-\frac{1}{2} + \frac{x^2}{3!} - \frac{x^4}{4!} + \dots\right)$

$$\boxed{= -\frac{1}{2}}$$

c) $\int_0^x e^{-t^2}\, dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right) dt$

$= t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \dots \Big|_0^x$

$$\boxed{= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots}$$

Using first two terms:

$\int_0^{1/2} e^{-t^2} = \left(\frac{1}{2}\right) - \frac{(1/2)^3}{3}$

$$\boxed{= \frac{11}{24}}$$

d) error $= |\text{act} - \text{est}| < \frac{1}{200}$

$\left|.46128\ldots - \frac{11}{24}\right| < \frac{1}{200}$

this is $\int_0^{1/2} e^{-t^2}$

$.00295 < .005$ ✓

OR error = next term: $\frac{x^5}{10}$

@ $x = 1/2$ $\quad \frac{(1/2)^5}{10} < \frac{1}{200}$

$\frac{1}{320} < \frac{1}{200}$ ✓